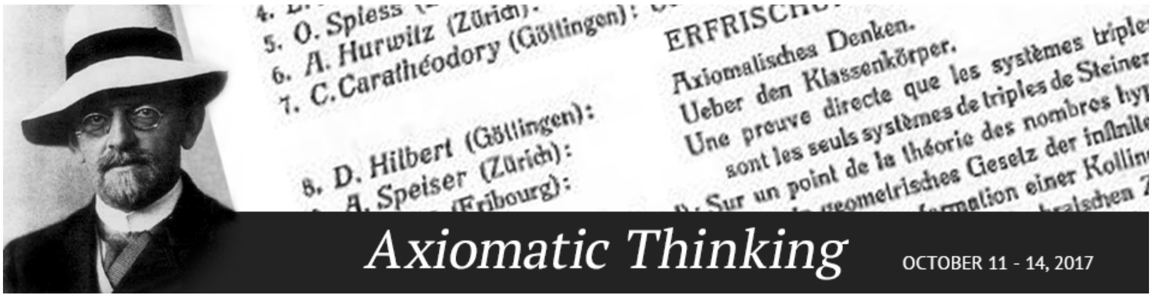
Axiomatic Thinking
OCTOBER 11 - 14, 2017

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Axiomatic Thinking

Joint conference of the *Academia das Ciências de Lisboa*
and the *Académie Internationale de Philosophie des Sciences*

Lisbon, October 11–14, 2017

`http://eventos.fct.unl.pt/aips17`

Programme

Wednesday, Oct. 11, 2017

9:00–13:00	Morning Session *Opening* Michael Detlefsen Wolfram Pohlers Steve Simpson
13:00–14:30	Lunch break
14:30–18:00	Afternoon Session Peter Koellner Michael Rathjen Mark van Atten
18:00–19:00	Panel Discussion
19:30	(Dinner)

Thursday, Oct. 12, 2017

8:30–12:30	Morning Session Evandro Agazzi José Ferreirós Wilfried Sieg
12:30–14:00	Lunch break
14:00–18:00	Afternoon Session Marco Buzzoni Gregor Schiemann Daniel Vanderveken Fabio Minazzi
18:30	*AIPS Assembly**

Friday, Oct. 13, 2017

8:30–12:30	Morning Session Michel Ghins Dennis Dieks Alexander Afriat & Gino Tarozzi
12:30–14:00	Lunch break
14:00–18:00	Afternoon Session Mario Alai Lorenzo Magnani Olga Pombo Valentin Bazhanov Alberto Cordero
20:00	Conference Dinner

Saturday, Oct. 14, 2017

8:30–12:30	Morning Session Paul Weingartner Itala D'Ottaviano Jean Petitot
12:30	Lunch
Afternoon	Lisbon Excursion

On Wednesday the meeting will take place at the Academia das Ciências de Lisboa:
R. da Academia das Ciências, 19 1249-122 Lisboa. `http://www.acad-ciencias.pt`

Thursday to Saturday the meeting will take place in the Auditorium of the Library, FCT, Universidade Nova de Lisboa: Campus de Caparica, 2829-516 Caparica. `http://www.fct.unl.pt`

All participants will stay at the Hotel TRYP Lisboa Caparica Mar: Av. Gen. Humberto Delgado 47, 2829-506 Costa da Caparica. `http://www.tryplisboacaparica.com` (Transport to the conference place will be organized.)

*The Assembly is only for full members of the AIPS.

Contents

Can some proposed extensions of Hilbert's programme to the foundations of physics, like quantum logic and quantum probability, really contribute to the solution of quantum paradoxes?

Alexander Afriat & Gino Tarozzi

For almost a century, impressive efforts have been devoted the well-known foundational problems of quantum mechanics. The efforts have been made in different directions, giving rise to the formation of various foundational sectors of research. Each sector has developed its own jargon, idiosyncrasies, and especially certainties: devout votaries have even been known to insist that their approach, unlike the others, solves all the problems of quantum mechanics. What is true is that the various approaches are not pointless, each offering its own perspective, which can shed light from a particular angle.

Quantum logic and quantum probability in their effort to provide strictly formal solutions of quantum paradoxes, have been represented as axiomatisations in the spirit of Hilberts programme. We can consider them as such, to assess their contributions to the logical clarification of some crucial points in the foundational debate.

The semantic function of the axiomatic method

Evandro Agazzi

The origin of the axiomatic method derived, in ancient Greek philosophy, from the requirement of providing a logical justification for the truth of a statement that, in such a way, passed from the status of belief to the status of knowledge. This justification was conceived as a logical deduction from first principles endowed with intrinsic evidence and this, in turn, was supposed to consist in an intellectual intuition of the essential properties of the entities about which the discourse of a given science is organized. The 'crisis' of the mathematical intuition occurred at the end of the 19th century showed the weakness of such a view and opened the way to the formalistic approach inaugurated by Peano's school and strongly developed by Hilbert and his school. The axioms (or postulates) were no longer considered as statements expressing the properties of independently existing objects, but as expressing the conditions for the existence of possible domains of objects. This happens because the axioms constitute a kind of 'global' definition of the meaning of all the terms occurring in their formulation. This gives to the axioms a genuine semantic function concerning the sense of the respective concepts. This, however, does not provide them with a genuine referential function (that is, the capacity of 'creating' mathematical objects) contrary to what is often maintained.

Scientific realism and the empirical underdetermination of theories

Mario Alai

The empirical underderdetermination argument against scientific realism rests on a logically very strong point, but it seems to play little or no role in actual scientific practice and does not show up in the history of science. It is incumbent on scientific realists to explain why.

The most plausible answer is that a crucial premise of the argument, that all the evidence for a theory consists in its empirical consequences, is not accepted by scientists, nor should it be accepted by philosophers. In fact, not all empirical consequences confirm, moreover confirmation for a theory can come from (a) data which are not its empirical consequences and (b) non empirical facts and other theories.

This is so because scientists don't search just for theories that "save the phenomena", but for true theories, and they take theoretical virtues (explanatory power, fecundity, plausibility in the light of observable mechanisms and accepted background theories, consilience, systematic power) to be reliable symptoms of true. Scientific realists have arguments to back up this conviction. Examples have been brought up of alternative theories which are underdetermined *even* by theoretical virtues. However

1. some are trivial translations of one and the same theory in a different language;
2. some are expressions of the same content in a different conceptual scheme or frame of reference;
3. the most interesting examples are some are mathematically intertranslatable theories, with whom realists can deal in three ways: (3.1) suggest that some discriminating empirical evidence can be found in the future; (3.2) hold that they can be decided by theoretical plausibility criteria; (3.3) suspend judgment about their diverging content, granting that there may be fact of the matter even about questions which are undecidable in fact or in principle.

Kyle Stanford argued with convincing historical examples that the possibility of unconceived alternatives to current theories produces a transient but recurrent underdetermination. While this is different from the classical underdetermination "by all empirical evidence", it threatens to produce the same consequences as the infamous pessimistic meta-induction, which his examples immediately elicit: there is no truth in past theories, hence also in present and future ones. Realist however can resist these conclusions by the same two strategies used against the pessimistic induction: the discontinuity strategy, stressing that scientific progress effectively reduces underdetermination; and deployment realism, arguing that novel predictions show that even discarded theories include some partial truth, and such truth cumulate over time.

Predicativity and parametric polymorphism of Brouwerian implication

Mark van Atten

A common objection to the definition of intuitionistic implication in the Proof Interpretation is that it is impredicative. After some brief remarks on the history of that objection, I will argue that in Brouwer's writings predicativity of implication is ensured through parametric polymorphism of functions on species.

Revolution in Social and Cultural Neuroscience: Impact on Epistemology and Philosophy of Science

Valentin A. Bazhanov

The paper presents an attempt to assess from the philosophical standpoint the advanced social and cultural neuroscience results. These results enables to claim that the traditional comprehension of subject of cognition due to be reconstructed. We must move from its universalistic interpretation mostly manifested in classical transcendentalism to interpretation explicitly taking into account socio-cultural context of subjects activity, and sometimes even its biological background.

The ideas of cultural dependence of neural networks activity of the brain, a chain of mutual influences of human genes and culture, the different neurocognitive strategies of so called collectivists and individualistic cultures and their carriers, cultural and cognitive neurobiological determination of subjects activity will be discussed.

Are There Thought Experiments in Mathematics?

Marco Buzzoni

The fast-growing literature on thought experiments has generated a large number of different views, but has paid insufficient attention to mathematical thought experiments. The attempt to extend thought experimentation from the natural sciences to mathematics succeeds for applied mathematics, but works only in a limited sense in the case of pure mathematics. Unlike empirical thought experiments in applied mathematics, in pure mathematics there is strictly speaking no distinction in principle between real experiments and thought experiments, because the anticipation in thought of the solution of a problem in pure mathematics amounts to its actual solution, leaving no room for a separate real performance of the experiment. Even though visualisation plays a certain role in the thought experiments of pure mathematics, mathematical thought experiments are more similar, in their epistemologically fundamental aspects, to formal proofs than to thought experiments in the natural sciences.

One could obviously say that thought experiments are so ubiquitous in mathematics that they are an integral part of it, but the fact remains that, to the extent

that the dialectical relation of contraposition and reciprocal implication of thought experiments and real world experiments vanishes, we cannot avoid the conclusion that thought experiments must play fundamentally different roles in pure mathematics and in the natural sciences.

Selectivism: An Approach Long in the Making

Alberto Cordero

According to selectivism, (a) theories are not monolithic proposals but intellectual constructs made of posits of various degrees of success with respect to truth, (b) empirically successful theories flourish because the world is as some theoretical explanations and narratives they posit say it is, and (c) recognizing this, scientists try to grade theory components accordingly, if with uneven results. Current selectivism (the divide et impera approach) arises most proximately from responses to Laudans pessimistic inductions from the history of science, but the approach is much older, or so I argue in this paper.

I trace selectivism to epistemological and methodological schemes on view since Antiquitynotably in such developments as Ptolemaic and Copernican astronomy; Galileo's piece-meal approach to the study of nature, also his efforts to embrace realism about both the Bible and the Heliocentric Theory; Newton's proposed reform of natural philosophy, and (at the apex of classical physics) Lorentz's reading of Maxwell's theory. These and numerous other cases, I suggest, show regular recognition by past scientists that successful theories contain both "wheat" and "chaff" that need to be detached from each other, attesting to a selectivist core at work during most of the history of natural philosophy and science. At each stage, this core together with local background knowledge guided gradation of intellectual content and preferred retention as science advanced. Until about the late Renaissance, the resulting rational gradations emphasized deductive reasoning and meta-empirical certitudes; retention of intellectual content was poor except at levels guarded against revision by metaphysical or religious convictions. Ptolemaic astronomy (which, contrary to popular opinion, embodied a partial realist stance) exemplifies this stage well. By contrast, when at the dawn of modernity natural philosophers began to challenge the content and character of traditional knowledge, the gradation strategy reoriented accordingly. I focus on some emblematic episodes: (a) Galileo (Dialogue, Discourses, also his Letters to Duchess Christina); (b) Newton (Principia, Opticks); and (c) ampliative strategies in the century of Fresnel, Wheewell, Maxwell, and Lorentz. Cases such as these, I suggest, show how and why the tenets of today's divide et impera selectivism arose. What counted as acceptable natural philosophy altered along the way, as did the selectivist emphasis, which increasingly shifted towards partial piece-meal descriptions and theories that provided incomplete understanding of their intended domains. It became satisfactory to pursue knowledge through less than apodictic or even deductive proof, a trend fortified by methods focused on recognizing inductive markers of truthful theories. In the early 19th century the markers of choice were parsimony and fruitfulness, predictive power gaining favor later in the century. Recognition of these inductive indicators has led to unprecedented quality and quantity retentions of theory-parts at inductive levels.

A complementary question arises, however: If selectivist schemes have been long in the background, why does selectivism seem new? The last section considers this issue and calls attention to the enduring impact of some views from the mid-twentieth century.

Hilbert on Axiomatic Thinking and Rigor

Michael Detlefsen

> To proceed axiomatically means nothing other than to think with awareness (mit Bewusstsein denken).
>
> Hilbert, 1922.

In this paper, I will clarify and explore what I take Hilbert to have been saying in this statement. I will pay particular attention to what I think are the implications concerning rigor.

In the late nineteenth century, Pasch made a well known statement concerning the conditions of attaining rigor in geometrical proof. The criterion he proposed was what I will call an abstractionist criterion. It called not only for the elimination of appeals to geometrical figures in geometrical proofs, but for the elimination of appeals to meanings of geometrical terms generally.

Not long after Pasch proposed his criterion, Hilbert proposed a somewhat different standard of rigor which was inspired by what he took to be the distinctive feature(s) of axiomatic thinking. In this paper, I will consider the relationship between Pasch's abstractionist standard of rigor and Hilbert's so-called axiomatic standard.

Going to the Essence of Things: von Neumanns Axiomatization of Quantum Mechanics

Dennis Dieks

Von Neumanns 1932 book "Mathematische Grundlagen der Quantenmechanik" (Mathematical Foundations of Quantum Mechanics) is generally considered to be a milestone in the history of quantum mechanics, because of its unification of wave and matrix mechanics, its introduction of new statistical methods, its famous/notorious impossibility proof for "hidden variables", and several other contributions. What has not been generally appreciated is that the book's core consists in an application of the axiomatic method. In the talk we will investigate the nature of von Neumanns axiomatization of quantum mechanics and analyze how this axiomatization served to achieve the book's goals. We will conclude with some more general remarks about the usefulness of axiomatizing physical theories.

The two sides of axiomatics

José Ferreirós

In this talk we shall go beyond the merely syntactic foundational approach to axiomatics that was emphasized in the 1930s and can still be found e.g. in Bourbaki's Elèments de mathèmatique, vol. 1 (Set theory). In this connection one may contrast the formal-logical facet of axiomatic work, with the conceptual-mathematical facet; Hilbert's paper 'Axiomatischen Denken' in effect insists more on the latter than the former. That is in contrast with his subsequent work of the 1920s.

An adequate understanding of both aspects is crucial to properly see the role of axiomatic thinking in mathematics, but many philosophers and logicians have tended to emphasize only the first facet. In the talk, we shall consider some relevant historical examples of this duality, which is also related with the question how to properly understand the structuralist methodology of modern math; here again, one must go beyond what is typically found in philosophical work, or even in Bourbaki's famous 1950 paper on 'The architecture of modern mathematics'.

An attempt will be made to offer an elementary example of the contrast in question, between the two facets, through the differences between the axiomatizations of arithmetic offered by Peano in 1889 and Dedekind in 1888.

Axiomatized systems, scientific laws and laws of nature

Michel Ghins

David Lewis famously proposed a neo-Humean account of laws according to which the laws are the universal statements which figure as axioms or theorems in the scientific axiomatic systems which realize the best balance between simplicity and empirical force. Such account is known as the Mill-Ramsey-Lewis (MRL) view of laws. The merits and weaknesses of this approach have been widely discussed in the literature. I will argue that the emphasis laid on scientific theories is the main merit of the MRL axiomatic approach. Irrespective of the axiomatizability of a scientific theory, it is possible to identify the *scientific* laws as universal statements expressing regularities, which belong to a scientific theory. The nomicity of a scientific law thus rests on its belonging to a successful scientific theory. Such criterion is faithful to the spirit of neo-Humean approaches to laws and permits to solve the problem of inference and the problem of identification. However, this account fails to satisfy two important requirements for an adequate account of laws, namely the explanation of the existence of regularities and the fact that laws support the truth of counterfactual conditionals. It will be argued that in order to satisfy these requirements we must resort to a thin metaphysics of causal powers which ground the nomicity of universal statements which are not only scientific laws, but also laws of *nature*.

Two Futures: Pattern and Chaos

Peter Koellner

Set theory is presently at a cross roads, where one is faced with two radically different possible futures.

This is first indicated by Woodin's HOD Dichotomy Theorem, an analogue of Jensen's Covering Lemma with HOD in place of L. The HOD Dichotomy Theorem states that if there is an extendible cardinal, δ, then either HOD is "close" to V (in the sense that it correctly computes successors of singular cardinals greater than δ) or HOD is "far" from V (in the sense that all regular cardinals greater than or equal to δ are measurable in HOD). The question is whether the future will lead to the first or the second side of the dichotomy. Is HOD "close" to V, or "far" from V?

There are two opposing research programs leading to opposite sides of the dichotomy. The first program is the program of inner model theory. In recent years Woodin has shown that if inner model theory can reach one supercompact cardinal then it "goes all the way," and he has formulated a precise conjecture — the Ultimate-L Conjecture — which, if true, would lead to a fine-structural inner model that can accommodate all of the standard large cardinals. This is the future where pattern prevails.

The second program is the program of large cardinals beyond choice. Kunen famously showed that if AC holds then there cannot be a Reinhardt cardinal. It has remained open whether Reinhardt cardinals are consistent in ZF alone. In recent work — joint with Bagaria and Woodin — the hierarchy of large cardinals beyond choice has been investigated. It turns out that there is an entire hierarchy of choiceless large cardinals of which Reinhardt cardinals are only the beginning, and, surprisingly, this hierarchy appears to be highly ordered and amenable to systematic investigation. Perhaps it is even consistent.... The point is that if these choiceless large cardinals are consistent then the Ultimate-L Conjecture must fail. This is the future where there can be no fine-structural understanding of the standard large cardinals. This is the future where chaos prevails.

Contradiction, consistency and the paraconsistent perspective in the Western thought: from Heraclitus to Newton da Costa

Itala M. Loffredo D'Ottaviano

In this presentation I will outline a historical analysis on how a paraconsistent perspective was properly constituted in the Western thought and how principles, rules and axiomatic logical systems begin to express distinct concepts of paraconsistency.

By analyzing the historical precedents of paraconsistent logic before the 20th century, we can identify some unanswered questions:

- What ideas were proposed and debated with regard to consistency in that period of the history of formal logic?
- Did such ideas influence later logical theories?
- Was there knowledge of logical rules and principles which allowed, in some contexts, for inconsistency to be dealt without trivialization?
- If such principles were known, how were these proto-principles stated, and in what way can they be related to the logical-paraconsistent results and rules known today?

Knowledge in motion. How knowledge transfer in science affects eco-cognitive openness, creativity, and epistemic responsibility

Lorenzo Magnani

Taking advantage of logical and cognitive research in abductive reasoning, which emphasizes the crucial role played by the so-called "maximization of eco-cognitive openness", I will illustrate the importance of knowledge transfer — knowledge in motion — in multidisciplinary, interdisciplinary, and transdisciplinary scientific research. Further, the hot problem of the current emergence of various kinds of "epistemic irresponsibility" will be introduced and some cases related to the commercialization of science, when knowledge transfer is jeopardized and creativity endangered, described.

Axiomatic Thinking and Philosophy — From neo-scholastics logic to neo-realism

Fabio Minazzi

Can you mathematize logic without reducing it to a formal calculation? This is the problem of Frege. But this is also the philosophical problem of a *formal* logic soon turned into a *formalistic* conception of logic. There is a *semantic function* of the axiomatic method? The axioms of a theory can be thought of as set out "meaningless"?Or are these axioms continue to maintain a *meaning* even if they lack a *referent*?

Just philosophical discussions of medieval scholasticism help to better understand the problem of the *semantic function* of the axiomatic method. Which it helps to put in a new perspective the relationship between formalization and apophantic logorecovering the relationship between *Sinn* and *Bedeutung* elaborated by Frege. The neo-logical realism is thus a philosophical perspective that can envisage a new relationship between the semantic logo and apophantic logo, opening to an interesting and new epistemological perspective.

Axiomatic as a strategy for complex proofs: the case of Riemann Hypothesis

Jean Petitot

My purpose is to comment some claims of André Weil (1906-1998) in his letter of March 26, 1940 to his sister Simone: "it is essential, if mathematics is to stay as a whole, to provide a unification, which absorbs in some simple and general theories all the common substrata of the diverse branches of the science, suppressing what is not so useful and necessary, and leaving intact what is truly the specific detail of each big problem. This is the good one can achieve with axiomatics." For Weil (and Bourbaki) the main problem was to find "strategies" for inventing complex proofs of "big problems". For that, the dialectic balance between general structures and specific details is crucial. I will focus on the fact that, for these creative mathematicians, the concept of structure is a functional concept, which has always a "strategic" creative function.

I will take the case of Artin, Schmidt, Hasse and Weil who introduced an intermediary third world between, on the one hand, Riemann original hypothesis on the non trivial zeroes of the zeta function in analytic theory of numbers, and, on the other hand, the algebraic theory of compact Riemann surfaces. The intermediary world is that of projective curves over finite fields of characteristic $p \geq 2$. RH can be translated in this context and can be proved using sophisticated tools of algebraic geometry (divisors, Riemann-Roch theorem, intersection theory, Severi-Castelnuovo inequality) coupled with the action of Frobenius maps in characteristic $p \geq 2$. Recently, Alain Connes proposed a new strategy and constructed a new topos theoretic framework *à la* Grothendieck were Weil's proof could be transferred by analogy back to the original RH. The fundamental discovery is that one can work in the world of "tropical algebraic geometry in characteristic 1" and "idempotent analysis".

On the performance of axiom systems

Wolfram Pohlers

We introduce an ordinal $\delta^{\mathfrak{M}}(\mathsf{T})$ as a measure for the performance of an axiom system T for a countable acceptable structure $\mathfrak{M}$. It will turn out that $\delta^{\mathfrak{M}}(\mathsf{T})$ rather measures the performance of the axiom system in respect to an universe above $\mathfrak{M}$. If T axiomatizes even a hierarchy of universes above $\mathfrak{M}$ the ordinal $\delta^{\mathfrak{M}}(\mathsf{T})$ turns into a family $\mathit{Spec}^{\mathfrak{M}}(\mathsf{T})$ of ordinals, the *spectrum of* T. In the end we will indicate (if time allows) how to fill in the lacking point below ω in $\mathit{Spec}^{\mathfrak{M}}(\mathsf{T})$ which also measures the performance of T in respect to $\mathfrak{M}$.

The seriousness of the arbitrary claim

Olga Pombo

Arbitrary claim is always underlying the question of convention. And it has serious implications. What I propose is to examine the question on basis of the analysis of an example in which this issue was crucial, even dangerous.

The example concerns Thomas Hobbes defense of the arbitrariness of language, namely the analysis of the arguments and counterarguments he was obliged to put forward against those who, in his time, were claiming for the opposite thesis.

Type theories, (intuitionistic) set theories and univalence

Michael Rathjen

Type theory, originally conceived as a bulwark against the paradoxes of naive set theory, has languished for a long time in the shadow of axiomatic set theory which became the mainstream foundation of mathematics in the 20th century. But type theories, especially dependent ones *à la* Martin-Löf, are looked upon favorably these days. The recent renaissance not only champions type theory as a central framework for constructive mathematics and as an important tool for achieving the goal of fully formalized mathematics (amenable to verification by computer-based proof assistants) but also finds deep and unexpected connections between type theory and other areas of mathematics (via homotopy type theory). One aspect, though, that makes type theories irksome is their overbearing syntax and rigidity. It is probably less well-known that they can often be related to set theories, albeit intuitionistic ones, and thereby rendered more accessible to those who favor breathing “set-theoretic” air.

Open Mathematization — On the Tension between Plurality and Unity of Scientific Knowledge in David Hilbert's "Axiomatic Thinking"

Gregor Schiemann

Hilbert's 1917 lecture "Axiomatic Thinking" formulates a program of mathematization of the sciences that has lost none of its relevance. It is situated within a field of tension between two extremes: On the one hand, Hilbert presents axiomatic thinking as a method which, as a formal procedure, can be applied to different scientific contents. Contrasted with this is, on the other hand, the quest for a universally valid equation from which all scientific knowledge can be deduced together with additional assumptions. Whatever form mathematization may assume, it represents a radical program of unifying scientific knowledge. Against this background, the recognition of the plurality of science also expressed in the lecture forms a remarkable contrast which calls for explanation.

Proofs as objects: a pivotal thought

Wilfried Sieg

It is a remarkable fact that Hilberts programmatic papers from the 1920s, almost exclusively, shape the contemporary perspective of his views concerning mathematics. Even his own quite different foundational work on geometry and arithmetic from the late 1890s is often understood from that vantage point. I am pursuing two goals, namely, (1) to contrast Hilberts *formal axiomatic* method from the early 1920s with his *structural axiomatic approach* from the 1890s, and (2) to emphasize how the two approaches can be fruitfully joined for the beginnings of a theory of mathematical proofs. The development toward such a *theory* began in 1917 when Hilbert gave his talk *Axiomatisches Denken*. Hilbert suggested, in particular:

> [...] we must — that is my conviction — turn the concept of the specifically mathematical proof into an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

Hilbert recognized in the next sentence that "the execution of this program is at present, to be sure, still an unsolved task". Dramatic steps in the pursuit of that program have been taken since and much fascinating work remains to be done: we have excellent reasons to celebrate the 100^{th} anniversary of Hilbert's talk.

Foundations of mathematics: an optimistic message

Stephen Simpson

Historically, mathematics has often been regarded as a role model for all of science — a paragon of abstraction, logical precision, and objectivity. The 19th and early 20th centuries saw tremendous progress. The great mathematician David Hilbert proposed a sweeping program whereby the entire panorama of higher mathematical abstractions would be justified objectively and logically, in terms of finite processes. But then in 1931 the great logician Kurt Gödel published his famous incompleteness theorems, leading to an era of confusion and skepticism. In this talk I show how modern foundational research has opened a new path toward objectivity and optimism in mathematics.

A Simple Formulation of the Logic of First Level Attitudes, Actions and Illocutions

Daniel Vanderveken

As G. Frege (1918) and J.L. Austin (1956) pointed out, the primary units of meaning and communication in the use and comprehension of language are illocutionary acts with felicity conditions rather than propositions with truth conditions. In *Foundations of Illocutionary Logic* (1985) J.R. Searle and I stated the principles of the theory of felicity of first level illocutionary acts attempted by individual speakers at a single moment of utterance. Later in *Meaning and Speech Acts* (1990-91), I used the resources of proof- and model-theory in order to formulate the logic of elementary illocutionary acts with a *force* and a propositional content. I presented a sound and generally complete axiomatization of necessary and universal valid laws of felicity governing elementary illocutionary acts. I have recently revised in 2008 and 2009 the standard logic of attitudes of J. Hintikka (1962, 1971) in order to deal with all psychological modes and to account for the fact that human agents are neither logically omniscient nor perfectly rational. I have also revised in 2005 and 2014 the logic of action of N. Belnap (1992, 2002) in order to explicate the nature of basic and intentional actions and of action generation.

I will now formulate a richer first level illocutionary logic that contains a new logic of ramified time and of first level attitudes and actions. We live in an indeterminist world with an open future. Thanks to ramified time I will account for the freedom of agents and better analyze satisfaction conditions of illocutions and attitudes whose propositional content is future. Our illocutionary acts are intrinsically intentional actions. Thanks to the new logic of attitudes and actions I will account for the intentionality and minimal rationality of human speakers and the generation of different kinds of speech acts (like acts of utterance, of presupposition, of expression of propositional contents and attitudes, illocutionary and perlocutionary acts) in the context of meaningful utterances. I will first present the rules of formation and of abbreviation of my ideal object-language. Next I will define its model-theoretical semantics and I will formulate basic axioms and enumerate fundamental valid laws.

Axiomatic Thinking – Applied to Religion

Paul Weingartner

As is known Bochenski was the first to apply axiomatic thinking to religion in a broader sense. First in his Logic of Religion (1965) and then in a stricter sense in his formalization of Aquina's 5 ways and still in more detail in his Gottes Dasein und Wesen (2003). The present paper will start with some preliminaries for applying logic to religion. In a second part an axiomatic system will be presented which includes definitions of omniscience and omnipotence and of God's relation to creation including providence. With some suitable axioms and further definitions one can derive theorems about God's knowledge and God's will with respect to the world, to man and to occurring moral evil.

VIII. Ordentliche Sitzung

der

Schweizerischen Mathematischen Gesellschaft

gemeinsam mit der

99. Jahresversammlung der Schweiz. Naturforschenden Gesellschaft

in Zürich

Dienstag, den 11. September 1917, im Hörsaal 304 der Universität

Vormittags punkt 8 Uhr

Vorträge und Mitteilungen der Herren:

1. *A. Emch* (Urbana U. S. A.): Ueber ebene Kurven, welche die n. Einheitswurzeln in der Ebene zu reellen Brennpunkten haben.
2. *G. Pólya* (Zürich): Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen.
3. *F. Gonseth* (Zürich): Un théorème sur deux ellipsoides confocaux.
4. *L. Kollros* (Zürich): Propriétés métriques des courbes algébriques.
5. *O. Spiess* (Basel): Ein Satz über rationale Funktionen.
6. *A. Hurwitz* (Zürich): Verallgemeinerung des Pohlkeschen Satzes.
7. *C. Carathéodory* (Göttingen): Ueber die geometrische Behandlung der Extrema von Doppelintegralen.

ERFRISCHUNGSPAUSE.

8. *D. Hilbert* (Göttingen): Axiomatisches Denken.
9. *A. Speiser* (Zürich): Ueber den Klassenkörper.
10. *S. Bays* (Fribourg): Une preuve directe que les systèmes triples de Kirkman et de Netto sont les seuls systèmes de triples de Steiner existants pour 13 éléments.
11. *L. G. Du Pasquier* (Neuchâtel): Sur un point de la théorie des nombres hypercomplexes.
12. *H. Berliner* (Bern): Ueber ein geometrisches Gesetz der infiniten Pluralität.
13. *K. Merz* (Chur): Quadratische Transformation einer Kollineation.
14. *G. Pólya* (Zürich): Ganzwertige Polynome in algebraischen Zahlkörpern.
15. *L. G. Du Pasquier* (Neuchâtel): Une nouvelle formule d'interpolation dans la théorie mathématique de la population.

NB. Die Herren Vortragenden werden ersucht, für die Dauer ihrer Vorträge nicht mehr als 20 Minuten in Aussicht zu nehmen und dem Sekretär einen Auszug ihrer Mitteilungen noch vor Schluss der Tagung abzugeben.

Gemeinsames Mittagessen um 1 Uhr im Hotel „Pelikan"

Im Anschluss daran sollen die Vereinsgeschäfte erledigt werden:

Abnahme der Jahresrechnung und Neuwahl des Vorstandes.

An die Mitglieder der Schweiz. Mathematischen Gesellschaft!

Sehr geehrte Herren Kollegen, wir unterbreiten Ihnen die reichhaltige Tagesordnung unserer ordentlichen Jahresversammlung und bitten um zahlreiche Teilnahme an dieser Tagung, welche eine besondere wissenschaftliche Bedeutung erhält durch den Vortrag des Herrn Prof. Hilbert aus Göttingen, der auf eine Einladung des Vorstandes über eine wissenschaftliche Methode sprechen wird, die ihm eine ausschlaggebende Förderung verdankt.

Der Vorstand der Schweiz. Mathematischen Gesellschaft:

Der Präsident:
Prof. Dr. **M. Grossmann** (Zürich).

Der Vizepräsident:
Prof. Dr. **M. Plancherel** (Fribourg).

Der Sekretär:
Prof. Dr. **L. Crelier** (Biel-Bern).

Axiomatisches Denken.*)

Von

David Hilbert in Göttingen.

Wie im Leben der Völker das einzelne Volk nur dann gedeihen kann wenn es auch allen Nachbarvölkern gut geht, und wie das Interesse der Staaten es erheischt, daß nicht nur innerhalb jedes einzelnen Staates Ordnung herrsche, sondern auch die Beziehungen der Staaten unter sich gut geordnet werden müssen, so ist es auch im Leben der Wissenschaften. In richtiger Erkenntnis dessen haben die bedeutendsten Träger des mathematischen Gedankens stets großes Interesse an den Gesetzen und der Ordnung in den Nachbarwissenschaften bewiesen und vor allem zu Gunsten der Mathematik selbst von jeher die Beziehungen zu den Nachbarwissenschaften, insbesondere zu den großen Reichen der Physik und der Erkenntnistheorie gepflegt. Das Wesen dieser Beziehungen und der Grund ihrer Fruchtbarkeit, glaube ich, wird am besten deutlich, wenn ich Ihnen diejenige allgemeine Forschungsmethode schildere, die in der neueren Mathematik mehr und mehr zur Geltung zu kommen scheint: ich meine die *axiomatische Methode*.

Wenn wir die Tatsachen eines bestimmten mehr oder minder umfassenden Wissensgebietes zusammenstellen, so bemerken wir bald, daß diese Tatsachen einer Ordnung fähig sind. Diese Ordnung erfolgt jedesmal mit Hilfe eines gewissen *Fachwerkes von Begriffen* in der Weise, daß dem einzelnen Gegenstande des Wissensgebietes ein Begriff dieses Fachwerkes und jeder Tatsache innerhalb des Wissensgebietes eine logische Beziehung zwischen den Begriffen entspricht. Das Fachwerk der Begriffe ist nichts Anderes als die *Theorie* des Wissensgebietes.

So ordnen sich die geometrischen Tatsachen zu einer Geometrie, die arithmetischen Tatsachen zu einer Zahlentheorie, die statischen, mechanischen, elektrodynamischen Tatsachen zu einer Theorie der Statik, Mechanik, Elektrodynamik oder die Tatsachen aus der Physik der Gase zu einer Gastheorie. Ebenso ist es mit den Wissensgebieten der Thermodynamik, der geometrischen Optik, der elementaren Strahlungstheorie, der

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Wärmeleitung oder auch mit der Wahrscheinlichkeitsrechnung und der Mengenlehre. Ja es gilt von speziellen rein mathematischen Wissensgebieten wie Flächentheorie, Galoisscher Gleichungstheorie, Theorie der Primzahlen nicht weniger als für manche der Mathematik fern liegende Wissensgebiete wie gewisse Abschnitte der Psychophysik oder die Theorie des Geldes.

Wenn wir eine bestimmte Theorie näher betrachten, so erkennen wir allemal, daß der Konstruktion des Fachwerkes von Begriffen einige wenige ausgezeichnete Sätze des Wissensgebietes zugrunde liegen und diese dann allein ausreichen, um aus ihnen nach logischen Prinzipien das ganze Fachwerk aufzubauen.

So genügt in der Geometrie der Satz von der Linearität der Gleichung der Ebene und von der orthogonalen Transformation der Punktkoordinaten vollständig, um die ganze ausgedehnte Wissenschaft der Euklidischen Raumgeometrie allein durch die Mittel der Analysis zu gewinnen. Zum Aufbau der Zahlentheorie ferner reichen die Rechnungsgesetze und Regeln für ganze Zahlen aus. In der Statik übernimmt die gleiche Rolle der Satz vom Parallelogramm der Kräfte, in der Mechanik etwa die Lagrangeschen Differentialgleichungen der Bewegung und in der Elektrodynamik die Maxwellschen Gleichungen mit Hinzunahme der Forderung der Starrheit und Ladung des Elektrons. Die Thermodynamik läßt sich vollständig auf den Begriff der Energiefunktion und die Definition von Temperatur und Druck als Ableitungen nach ihren Variabeln, Entropie und Volumen, aufbauen. Im Mittelpunkt der elementaren Strahlungstheorie steht der Kirchhoffsche Satz über die Beziehungen zwischen Emission und Absorption; in der Wahrscheinlichkeitsrechnung ist das Gaußsche Fehlergesetz, in der Gastheorie der Satz von der Entropie als negativem Logarithmus der Wahrscheinlichkeit des Zustandes, in der Flächentheorie die Darstellung des Bogenelementes durch die quadratische Differentialform, in der Gleichnugstheorie der Satz von der Wurzelexistenz, in der Theorie der Primzahlen der Satz von der Realität und Häufigkeit der Nullstellen der Riemannschen Funktion $\xi(t)$ der grundlegende Satz.

Diese grundlegenden Sätze können von einem ersten Standpunkte aus als die *Axiome der einzelnen Wissensgebiete* angesehen werden: die fortschreitende Entwicklung des einzelnen Wissensgebietes beruht dann lediglich in dem weiteren logischen Ausbau des schon aufgeführten Fachwerkes der Begriffe. Zumal in der reinen Mathematik ist dieser Standpunkt der vorherrschende, und der entsprechenden Arbeitsweise verdanken wir die mächtige Entwicklung der Geometrie, der Arithmetik, der Funktionentheorie und der gesamten Analysis.

Somit hatte dann in den genannten Fällen das Problem der Begründung der einzelnen Wissensgebiete eine Lösung gefunden; diese Lösung war aber nur eine vorläufige. In der Tat machte sich in den einzelnen Wissensgebieten das Bedürfnis geltend, die genannten, als Axiome angesehenen und zugrunde gelegten Sätze selbst zu begründen. So gelangte man zu „Beweisen“ für die Linearität der Gleichung der Ebene und die Orthogonalität der eine Bewegung ausdrückenden Transformation, ferner für die arithmetischen Rechnungsgesetze, für das Parallelogramm der Kräfte, für die Lagrangeschen Bewegungsgleichungen und das Kirchhoffsche Gesetz über Emission und Absorption, für den Entropiesatz und den Satz von der Existenz der Wurzeln einer Gleichung.

Aber die kritische Prüfung dieser „Beweise“ läßt erkennen, daß sie nicht an sich Beweise sind, sondern im Grunde nur die Zurückführung auf gewisse tiefer liegende Sätze ermöglichen, die nunmehr ihrerseits an Stelle der zu beweisenden Sätze als neue Axiome anzusehen sind. So entstanden die eigentlichen heute sogenannten *Axiome* der Geometrie, der Arithmetik, der Statik, der Mechanik, der Strahlungstheorie oder der Thermodynamik. Diese Axiome bilden eine tiefer liegende Schicht von Axiomen gegenüber derjenigen Axiomschicht, wie sie durch die vorhin genannten zuerst zugrunde gelegten Sätze in den einzelnen Wissensgebieten charakterisiert worden ist. Das Verfahren der axiomatischen Methode, wie es hierin ausgesprochen liegt, kommt also einer *Tieferlegung der Fundamente* der einzelnen Wissensgebiete gleich, wie eine solche ja bei jedem Gebäude nötig wird in dem Maße, als man dasselbe ausbaut, höher führt und dennoch für seine Sicherheit bürgen will.

Soll die Theorie eines Wissensgebietes d. h. das sie darstellende Fachwerk der Begriffe ihrem Zwecke, nämlich der Orientierung und Ordnung dienen, so muß es vornehmlich gewissen zwei Anforderungen genügen: *erstens* soll es einen Überblick über die *Abhängigkeit* bzw. *Unabhängigkeit* der Sätze der Theorie und *zweitens* eine Gewähr der *Widerspruchslosigkeit* aller Sätze der Theorie bieten. Insbesondere sind die Axiome einer jeden Theorie nach diesen beiden Gesichtspunkten zu prüfen.

Beschäftigen wir uns zunächst mit der Abhängigkeit bzw. Unabhängigkeit der Axiome.

Das klassische Beispiel für die Prüfung der Unabhängigkeit eines Axioms bietet das *Parallelenaxiom* in der Geometrie. Die Frage, ob der Parallelensatz durch die anderen Axiome schon bedingt ist, verneinte Euklid, indem er ihn unter die Axiome setzte. Die Untersuchungsmethode Euklids wurde vorbildlich für die axiomatische Forschung, und seit Euklid ist zugleich die Geometrie das Musterbeispiel für eine axiomatisierte Wissenschaft überhaupt.

Ein anderes Beispiel für eine Untersuchung über die Abhängigkeit der Axiome bietet die klassische Mechanik. Vorläufigerweise konnten, wie vorhin bemerkt, die Lagrangeschen Gleichungen der Bewegung als Axiome der Mechanik gelten — läßt sich doch auf diese in ihrer allgemeinen Formulierung für beliebige Kräfte und beliebige Nebenbedingungen die Mechanik gewiß vollständig gründen. Bei näherer Untersuchung zeigt sich aber, daß beim Aufbau der Mechanik sowohl beliebige Kräfte wie beliebige Nebenbedingungen vorauszusetzen unnötig ist und somit das System von Voraussetzungen vermindert werden kann. Diese Erkenntnis führt einerseits zu dem Axiomensystem von Boltzmann, der nur Kräfte und zwar speziell Zentralkräfte, aber keine Nebenbedingungen annimmt und dem Axiomensystem von Hertz, der die Kräfte verwirft und mit Nebenbedingungen und zwar speziell mit festen Verbindungen auskommt. Diese beiden Axiomensysteme bilden somit eine tiefere Schicht in der fortschreitenden Axiomatisierung der Mechanik.

Nehmen wir bei Begründung der Galoisschen Gleichungstheorie die Existenz der Wurzeln einer Gleichung als Axiom an, so ist dieses sicher ein abhängiges Axiom; denn jener Existenzsatz ist aus den arithmetischen Axiomen beweisbar, wie zuerst Gauß gezeigt hat.

Ähnlich verhält es sich damit, wenn wir etwa den Satz von der Realität der Nullstellen der Riemannschen Funktion $\xi(t)$ in der Primzahlentheorie als Axiom annehmen wollten: beim Fortschreiten zur tieferen Schicht der reinen arithmetischen Axiome würde der Beweis dieses Realitätssatzes notwendig sein und dieser erst uns die Sicherheit der wichtigen Folgerungen gewähren, die wir durch seine Postulierung schon jetzt für die Theorie der Primzahlen aufgestellt haben.

Besonderes Interesse für die axiomatische Behandlung bietet die Frage der Abhängigkeit der Sätze eines Wissensgebietes von dem Axiom der *Stetigkeit*.

In der Theorie der reellen Zahlen wird gezeigt, daß das Axiom des Messens, das sogenannte Archimedische Axiom, von allen übrigen arithmetischen Axiomen unabhängig ist. Diese Erkenntnis ist bekanntlich für die Geometrie von wesentlicher Bedeutung, scheint mir aber auch für die Physik von prinzipiellem Interesse; denn sie führt uns zu folgendem Ergebnis: die Tatsache, daß wir durch Aneinanderfügen irdischer Entfernungen die Dimensionen und Entfernungen der Körper im Weltenraume erreichen, d. h. durch irdisches Maß die himmlischen Längen messen können, ebenso die Tatsache, daß sich die Distanzen im Atominneren durch das Metermaß ausdrücken lassen, sind keineswegs bloß eine logische Folge der Sätze über Dreieckskongruenzen und der geometrischen Konfiguration, sondern erst ein Forschungsresultat der Empirie. Die Gültigkeit des Archimedischen Axioms in der Natur bedarf eben im bezeichneten Sinne gerade so der Bestätigung durch das Experiment wie etwa der Satz von der Winkelsumme im Dreieck im bekannten Sinne.

Allgemein möchte ich das Stetigkeitsaxiom in der Physik wie folgt formulieren: „Wird für die Gültigkeit einer physikalischen Aussage irgend ein beliebiger Genauigkeitsgrad vorgeschrieben, so lassen sich kleine Bereiche angeben, innerhalb derer die für die Aussage gemachten Voraussetzungen frei variieren dürfen, ohne daß die Abweichung von der Aussage den vorgeschriebenen Genauigkeitsgrad überschreitet“. Dies Axiom bringt im Grunde nur zum Ausdruck, was unmittelbar im Wesen des Experimentes liegt; es ist stets von den Physikern angenommen worden, ohne daß es bisher besonders formuliert worden ist.

Wenn man z. B. nach Planck aus dem Axiom der Unmöglichkeit des *Perpetuum mobile zweiter Art* den zweiten Wärmesatz ableitet, so wird dabei dieses Stetigkeitsaxiom notwendigerweise benutzt.

Daß in der Begründung der Statik beim Beweise des Satzes vom *Parallelogramm der Kräfte* das Stetigkeitsaxiom notwendig ist — wenigstens bei einer gewissen nächstliegenden Auswahl der übrigen Axiome — hat Hamel auf eine sehr interessante Weise durch Heranziehung des Satzes von der Wohlordnungsfähigkeit des Kontinuums gezeigt.

Die Axiome der klassischen Mechanik können eine Tieferlegung erfahren, wenn man sich vermöge des Stetigkeitsaxioms die kontinuierliche Bewegung in kurz aufeinanderfolgende geradlinig gleichförmige stückweise durch Impulse hervorgerufene Bewegungen zerlegt denkt und dann als wesentliches mechanisches Axiom das *Bertrandsche Maximalprinzip* verwendet, demzufolge nach jedem Stoß die wirklich eintretende Bewegung stets diejenige ist, bei welcher die kinetische Energie des Systems ein Maximum wird gegenüber allen mit dem Satz von der Erhaltung der Energie verträglichen Bewegungen.

Auf die neuesten Begründungsarten der Physik, insbesondere der Elektrodynamik, die ganz und gar Kontinuumstheorien sind und demgemäß die Stetigkeitsforderung in weitestem Maße erheben, möchte ich hier nicht eingehen, weil diese Forschungen noch nicht genügend abgeschlossen sind.

Wir wollen nun den zweiten der vorhin genannten Gesichtspunkte, nämlich die Frage nach der *Widerspruchslosigkeit* der Axiome prüfen; diese ist offenbar von höchster Wichtigkeit, weil das Vorhandensein eines Widerspruches in einer Theorie offenbar den Bestand der ganzen Theorie gefährdet.

Die Erkenntnis der inneren Widerspruchslosigkeit ist selbst bei längst anerkannten und erfolgreichen Theorien mit Schwierigkeit verbunden: ich erinnere an den *Umkehr- und Wiederkehreinwand* in der kinetischen Gastheorie.

Oftmals passiert es, daß die innere Widerspruchslosigkeit einer Theorie als selbstverständlich angesehen wird, während in Wahrheit tiefe mathematische Entwicklungen zu dem Nachweise nötig sind. Als Beispiel betrachten wir ein Problem aus der elementaren Theorie der *Wärmeleitung*, nämlich die Temperaturverteilung innerhalb eines homogenen Körpers, dessen Oberfläche auf einer bestimmten von Ort zu Ort variierenden Temperatur gehalten wird: alsdann enthält in der Tat die Forderung des Bestehens von Temperaturgleichgewicht keinen inneren Widerspruch der Theorie. Zur Erkenntnis dessen ist aber der Nachweis nötig, daß die bekannte Randwertaufgabe der Potentialtheorie stets lösbar ist; denn erst die Lösung dieser Randwertaufgabe zeigt, daß eine der Wärmeleitungsgleichung genügende Temperaturverteilung überhaupt möglich ist.

Aber zumal in der Physik genügt es nicht, wenn die Sätze einer Theorie unter sich in Einklang stehen; vielmehr ist noch die Forderung zu erheben, daß sie auch den Sätzen eines benachbarten Wissensgebietes niemals widersprechen.

So liefern, wie ich kürzlich zeigte, die Axiome der elementaren Strahlungstheorie außer der Begründung des *Kirchhoffschen Satzes* über Emission und Absorption noch einen speziellen Satz über Reflexion und Brechung einzelner Lichtstrahlen, nämlich den Satz: Wenn zwei Strahlen natürlichen Lichtes und gleicher Energie von je einer Seite her auf die Trennungsfläche zweier Mediën in solchen Richtungen auffallen, daß der eine Strahl nach seinem Durchtritt, der andere nach seiner Reflexion dieselbe Richtung aufweist, so ist der durch die Vereinigung entstehende Strahl wieder von natürlichem Licht und gleicher Energie. Dieser Satz ist — wie sich in der Tat zeigt — mit der Optik keineswegs in Widerspruch, sondern kann als Folgerung aus der elektromagnetischen Lichttheorie abgeleitet werden.

Die Resultate der *kinetischen Gastheorie* stehen bekanntlich mit der *Thermodynamik* in bestem Einklang.

Ebenso sind *elektromagnetische Trägheit* und *Einsteinsche Gravitation* mit den entsprechenden Begriffen der klassischen Theorien verträglich, insofern diese letzteren als Grenzfälle der allgemeineren Begriffe in den neuen Theorien aufzufassen sind.

Dagegen hat die *moderne Quantentheorie* und die fortschreitende Erkenntnis der inneren Atomstruktur zu Gesetzen geführt, die der bisherigen wesentlich auf den Maxwellschen Gleichungen aufgebauten Elektrodynamik geradezu widersprechen; die heutige Elektrodynamik bedarf daher — wie jedermann anerkennt — notwendig einer neuen Grundlegung und wesentlichen Umgestaltung.

Wie man aus dem bisher Gesagten ersieht, wird in den physikalischen Theorien die Beseitigung sich einstellender Widersprüche stets durch veränderte Wahl der Axiome erfolgen müssen und die Schwierigkeit besteht darin, die Auswahl so zu treffen, daß alle beobachteten physikalischen Gesetze logische Folgen der ausgewählten Axiome sind.

Anders verhält es sich, wenn in rein theoretischen Wissensgebieten Widersprüche auftreten. Das klassische Beispiel für ein solches Vorkommnis bietet die Mengentheorie und zwar insbesondere das schon auf Cantor zurückgehende *Paradoxon der Menge aller Mengen.* Dieses Paradoxon ist so schwerwiegend, daß sehr angesehene Mathematiker, z. B. Kronecker und Poincaré, sich durch dasselbe veranlaßt fühlten, der gesamten Mengentheorie — einem der fruchtreichsten und kräftigsten Wissenszweige der Mathematik überhaupt — die Existenzberechtigung abzusprechen.

Auch bei dieser prekären Sachlage brachte die axiomatische Methode Abhilfe. Es gelang Zermelo, indem er durch Aufstellung geeigneter Axiome einerseits die Willkür der Definitionen von Mengen und andererseits die Zulässigkeit von Aussagen über ihre Elemente in bestimmter Weise beschränkte, die Mengentheorie derart zu entwickeln, daß die in Rede stehenden Widersprüche wegfallen, daß aber trotz der auferlegten Beschränkungen die Tragweite und Anwendungsfähigkeit der Mengentheorie die gleiche bleibt.

In allen bisherigen Fällen handelte es sich um Widersprüche, die sich im Verlauf der Entwicklung einer Theorie herausgestellt hatten und zu deren Beseitigung durch Umgestaltung des Axiomensystems die Not drängte. Aber es genügt nicht, vorhandene Widersprüche zu vermeiden, wenn der durch sie gefährdete Ruf der Mathematik als Muster strengster Wissenschaft wiederhergestellt werden soll: die prinzipielle Forderung der Axiomenlehre muß vielmehr weitergehen, nämlich dahin, zu erkennen, daß jedesmal innerhalb eines Wissensgebietes auf Grund des aufgestellten Axiomensystems Widersprüche *überhaupt unmöglich* sind.

Dieser Forderung entsprechend habe ich in den *Grundlagen der Geometrie* die Widerspruchslosigkeit der aufgestellten Axiome nachgewiesen, indem ich zeigte, daß jeder Widerspruch in den Folgerungen aus den geometrischen Axiomen notwendig auch in der Arithmetik des Systems der reellen Zahlen erkennbar sein müßte.

Auch für die physikalischen Wissensgebiete genügt es offenbar stets, die Frage der *inneren Widerspruchslosigkeit* auf die Widerspruchslosigkeit der arithmetischen Axiome zurückzuführen. So zeigte ich die Widerspruchslosigkeit der Axiome *der elementaren Strahlungstheorie*, indem ich für dieselbe das Axiomensystem aus analytisch unabhängigen Stücken aufbaute — die Widerspruchslosigkeit der Analysis dabei voraussetzend.

Ähnlich darf und soll man unter Umständen beim Aufbau einer mathematischen Theorie verfahren. Haben wir beispielsweise bei Entwicklung der Galoisschen Gruppentheorie den Satz von der *Wurzelexistenz* oder in der Theorie der Primzahlen den Satz von der *Realität der Nullstellen* der Riemannschen Funktion $\xi(t)$ als Axiom betrachtet, so läuft jedesmal der Nachweis der Widerspruchslosigkeit des Axiomensystems eben darauf hinaus, den Satz von der Wurzelexistenz bzw. den Riemannschen Satz über die Funktion $\xi(t)$ mit den Mitteln der Analysis zu beweisen — und damit erst ist die Vollendung der Theorie gesichert.

Auch die Frage der Widerspruchslosigkeit des Axiomensystems für die *reellen Zahlen* läßt sich auf die nämliche Frage für die ganzen Zahlen zurückführen: dies ist das Verdienst der Theorien der Irrationalzahlen von Weierstraß und Dedekind.

Nur in zwei Fällen nämlich, wenn es sich um die Axiome der *ganzen Zahlen* selbst und wenn es sich um die Begründung der *Mengenlehre* handelt, ist dieser Weg der Zurückführung auf ein anderes spezielleres Wissensgebiet offenbar nicht gangbar, weil es außer der Logik überhaupt keine Disziplin mehr gibt, auf die alsdann eine Berufung möglich wäre.

Da aber die Prüfung der Widerspruchslosigkeit eine unabweisbare Aufgabe ist, so scheint es nötig, die Logik selbst zu axiomatisieren und nachzuweisen, daß Zahlentheorie, sowie Mengenlehre nur Teile der Logik sind.

Dieser Weg, seit langem vorbereitet — nicht zum mindesten durch die tiefgehenden Untersuchungen von Frege — ist schließlich am erfolgreichsten durch den scharfsinnigen Mathematiker und Logiker Russell eingeschlagen worden. In der Vollendung dieses großzügigen Russellschen Unternehmens der *Axiomatisierung der Logik* könnte man die Krönung des Werkes der Axiomatisierung überhaupt erblicken.

Diese Vollendung wird indessen noch neuer und vielseitiger Arbeit bedürfen. Bei näherer Überlegung erkennen wir nämlich bald, daß die Frage der Widerspruchslosigkeit bei den ganzen Zahlen und Mengen nicht eine für sich alleinstehende ist, sondern einem großen Bereiche schwierigster erkenntnistheoretischer Fragen von spezifisch mathematischer Färbung angehört: ich nenne, um diesen Bereich von Fragen kurz zu charakterisieren, das Problem der prinzipiellen *Lösbarkeit einer jeden mathematischen Frage*, das Problem der nachträglichen *Kontrollierbarkeit* des Resultates einer mathematischen Untersuchung, ferner die Frage nach einem *Kriterium für die Einfachheit* von mathematischen Beweisen, die Frage nach dem

Verhältnis zwischen *Inhaltlichkeit und Formalismus* in Mathematik und Logik und endlich das Problem der *Entscheidbarkeit* einer mathematischen Frage durch eine endliche Anzahl von Operationen.

Wir können uns nun nicht eher mit der Axiomatisierung der Logik zufrieden geben, als bis alle Fragen dieser Art in ihrem Zusammenhange verstanden und aufgeklärt sind.

Unter den genannten Fragen ist die letzte, nämlich die Frage nach der Entscheidbarkeit durch eine endliche Anzahl von Operationen die bekannteste und die am häufigsten diskutierte, weil sie das Wesen des mathematischen Denkens tief berührt.

Ich möchte das Interesse für sie zu vermehren suchen, indem ich auf einige speziellere mathematische Probleme hinweise, in denen sie eine Rolle spielt.

In der Theorie der *algebraischen Invarianten* gilt bekanntlich der Fundamentalsatz, daß es stets eine endliche Anzahl von ganzen rationalen Invarianten gibt, durch die sich alle übrigen solchen Invarianten in ganzer rationaler Weise darstellen lassen. Der erste von mir angegebene allgemeine Beweis für diesen Satz befriedigt, wie ich glaube, unsere Ansprüche, was Einfachheit und Durchsichtigkeit anlangt, vollauf; es ist aber unmöglich, diesen Beweis so umzugestalten, daß wir durch ihn eine angebbare Grenze für die Anzahl der endlich vielen Invarianten des vollen Systems erhalten oder gar zur wirklichen Aufstellung derselben gelangen. Es sind vielmehr ganz anders geartete Überlegungen und neue Prinzipien notwendig gewesen, um zu erkennen, daß die Aufstellung des vollen Invariantensystems lediglich Operationen erfordert, deren Anzahl endlich ist und unterhalb einer vor der Rechnung angebbaren Grenze liegt.

Das gleiche Vorkommnis bemerken wir an einem Beispiel aus der *Flächentheorie*. In der Geometrie der Flächen vierter Ordnung ist es eine fundamentale Frage, aus wie vielen von einander getrennten Mänteln eine solche Fläche höchstens bestehen kann.

Das Erste bei der Beantwortung dieser Frage ist der Nachweis, daß die Anzahl der Flächenmäntel endlich sein muß; dieser kann leicht auf funktionentheoretischem Wege, wie folgt, geschehen. Man nehme das Vorhandensein unendlich vieler Mäntel an und wähle dann innerhalb eines jeden durch einen Mantel begrenzten Raumteiles je einen Punkt aus. Eine Verdichtungsstelle dieser unendlich vielen ausgewählten Punkte würde dann ein Punkt von solcher Singularität sein, wie sie für eine algebraische Fläche ausgeschlossen ist.

Dieser funktionentheoretische Weg führt auf keine Weise zu einer oberen Grenze für die Anzahl der Flächenmäntel; dazu bedarf es vielmehr gewisser Überlegungen über Schnittpunktsanzahlen, die dann schließlich lehren, daß die Anzahl der Mäntel gewiß nicht größer als 12 sein kann.

Die zweite von der ersten gänzlich verschiedene Methode läßt sich ihrerseits nicht dazu anwenden und auch nicht so umgestalten, daß sie die Entscheidung ermöglicht, ob eine Fläche 4^{ter} Ordnung mit 12 Mänteln wirklich existiert.

Da eine quaternäre Form 4^{ter} Ordnung 35 homogene Koeffizienten besitzt, so können wir uns eine bestimmte Fläche 4^{ter} Ordnung durch einen Punkt im 34-dimensionalen Raume veranschaulichen. Die Diskriminante der quaternären Form 4^{ter} Ordnung ist vom Grade 108 in den Koeffizienten derselben; gleich Null gesetzt, stellt sie demnach im 34-dimensionalen Raume eine Fläche 108^{ter} Ordnung dar. Da die Koffizienten der Diskriminante selbst bestimmte ganze Zahlen sind, so läßt sich der topologische Charakter der Diskriminantenfläche nach den Regeln, die uns für den 2- und 3-dimensionalen Raum geläufig sind, genau feststellen, so daß wir über die Natur und Bedeutung der einzelnen Teilgebiete, in die die Diskriminantenfläche den 34-dimensionalen Raum zerlegt, genaue Auskunft erhalten können. Nun besitzen die durch Punkte des nämlichen Teilgebietes dargestellten Flächen 4^{ter} Ordnung gewiß alle die gleiche Mäntelzahl, und es ist daher möglich, durch eine endliche, wenn auch sehr mühsame und langwierige Rechnung, festzustellen, ob eine Fläche 4^{ter} Ordnung mit $n \leqq 12$ Mänteln vorhanden ist oder nicht.

Die eben angestellte geometrische Betrachtung ist also ein dritter Weg zur Behandlung unserer Frage nach der Höchstzahl der Mäntel einer Fläche 4^{ter} Ordnung. Sie beweist die Entscheidbarkeit dieser Frage durch eine endliche Anzahl von Operationen. Prinzipiell ist damit eine bedeutende Förderung unseres Problems erreicht: dasselbe ist zurückgeführt auf ein Problem von dem Range etwa der Aufgabe, die $10^{(10^{10})}$te Ziffer der Dezimalbruchentwicklung von π zu ermitteln — einer Aufgabe, deren Lösbarkeit offenbar ist, deren Lösung aber unbekannt bleibt.

Vielmehr bedurfte es einer von Rohn ausgeführten tiefgehenden schwierigen algebraisch-geometrischen Untersuchung, um einzusehen, daß bei einer Fläche 4^{ter} Ordnung 11 Mäntel nicht möglich sind; 10 Mäntel dagegen kommen wirklich vor. Erst diese vierte Methode bringt somit die völlige Lösung des Problems.

Diese speziellen Ausführungen zeigen, wie verschiedenartige Beweismethoden auf dasselbe Problem anwendbar sind, und sollen nahelegen, wie notwendig es ist, das Wesen des mathematischen Beweises an sich zu studieren, wenn man solche Fragen, wie die nach der Entscheidbarkeit durch endlich viele Operationen mit Erfolg aufklären will.

Alle solchen prinzipiellen Fragen, wie ich sie vorhin charakterisierte und unter denen die eben behandelte Frage nach der Entscheidbarkeit durch endlich viele Operationen nur die letztgenannte war, scheinen mir ein wichtiges, neu zu erschließendes Forschungsfeld zu bilden, und zur Eroberung dieses Feldes müssen wir — das ist meine Überzeugung — den Begriff des spezifisch mathematischen Beweises selbst zum Gegenstand einer Untersuchung machen, gerade wie ja auch der Astronom die Bewegung seines Standortes berücksichtigen, der Physiker sich um die Theorie seines Apparates kümmern muß und der Philosoph die Vernunft selbst kritisiert.

Die Durchführung dieses Programms ist freilich gegenwärtig noch eine ungelöste Aufgabe.

Zum Schlusse möchte ich in einigen Sätzen meine allgemeine Auffassung vom Wesen der axiomatischen Methode zusammenfassen.

Ich glaube: Alles, was Gegenstand des wissenschaftlichen Denkens überhaupt sein kann, verfällt, sobald es zur Bildung einer Theorie reif ist, der axiomatischen Methode und damit mittelbar der Mathematik. Durch Vordringen zu immer tieferliegender Schichten von Axiomen im vorhin dargelegten Sinne gewinnen wir auch in das Wesen des wissenschaftlichen Denkens selbst immer tiefere Einblicke und werden uns der Einheit unseres Wissens immer mehr bewußt. In dem Zeichen der axiomatischen Methode erscheint die Mathematik berufen zu einer führenden Rolle in der Wissenschaft überhaupt.

‖1‖ Just as in the life of nations the individual nation can only thrive when all neighbouring nations are in good health; and just as the interest of states demands, not only that order prevail within every individual state, but also that the relationships of the states among themselves be in good order; so it is in the life of the sciences. In due recognition of this fact the most important bearers of mathematical thought have always evinced great interest in the laws and the structure of the neighbouring sciences; above all for the benefit of mathematics itself they have always cultivated the relations to the neighbouring sciences, especially to the great empires of physics and epistemology. I believe that the essence of these relations, and the reason for their fruitfulness, will appear most clearly if I describe for you the general method of research which seems to be coming more and more into its own in modern mathematics: I mean the *axiomatic method*.

‖2‖ When we assemble the facts of a definite, more-or-less comprehensive field of knowledge, we soon notice that these facts are capable of being ordered. This ordering always comes about with the help of a certain *framework of concepts* ‖*Fachwerk von Begriffen*‖ in the following way: a concept of this framework corresponds to each individual object of the field of knowledge, and a logical relation between concepts corresponds to every fact within the field of knowledge. The framework of concepts is nothing other than the *theory* of the field of knowledge.

[3] Thus the facts of geometry order themselves into a geometry, the facts of arithmetic into a theory of numbers, the facts of statics, mechanics, electrodynamics into a theory of statics, mechanics, electrodynamics, or the facts from the physics of gases into a theory of gases. It is precisely the same with the fields of knowledge of thermodynamics, geometrical optics, elementary radiation-theory, the conduction of heat, or also with the calculus of probabilities or the theory of sets. It even holds of special fields of knowledge in pure mathematics, such as the theory of surfaces, the theory of Galois equations, and the theory of prime numbers, no less than for several fields of knowledge that lie far from mathematics, such as certain parts of psychophysics or the theory of money.

[4] If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework.

[5] Thus in geometry the proposition of the linearity of the equation of the plane and of the orthogonal transformation of point-coordinates is completely adequate to produce the whole broad science of spatial Euclidean geometry purely by means of analysis. Moreover, the laws of calculation and the rules for integers suffice for the construction of number theory. In statics the same role is played by the proposition of the parallelogram of forces; in mechanics, say, by the Lagrangian differential equations of motion; and in electrodynamics by the Maxwell equations together with the requirement of the rigidity and charge of the electron. Thermodynamics can be completely built up from the concept of energy function and the definition of temperature and pressure as derivatives of its variables, entropy and volume. At the heart of the elementary theory of radiation is Kirchhoff's theorem on the relationships between emission and absorption; in the calculus of probabilities the Gaussian law of errors is the fundamental proposition; in the theory of gases, the proposition that entropy is the negative logarithm of the probability of the state; in the theory of surfaces, the representation of the element of arc by the quadratic differential form; in the theory of equations, the proposition concerning the existence of roots; in the theory of prime numbers, the proposition concerning the reality and frequency of Rieman's function $\zeta(s)$.

[6] These fundamental propositions can be regarded from an initial standpoint as the *axioms of the individual fields of knowledge*: the progressive development of the individual field of knowledge then lies solely in the further logical construction of the already mentioned framework of concepts. This standpoint is especially predominant in pure mathematics, and to the corresponding manner of working we owe the mighty development of geometry, of arithmetic, of the theory of functions, and of the whole of analysis.

[7] Thus in the cases mentioned above the problem of grounding the

individual field of knowledge had found a solution; but this solution was only temporary. In fact, in the individual fields of knowledge the need arose to ground the fundamental axiomatic propositions themselves. So one acquired 'proofs' of the linearity of the equation of the plane and the orthogonality of the transformation expressing a movement, of the laws of arithmetical calculation, of the parallelogram of forces, of the Lagrangian equations of motion, of Kirchhoff's law regarding emission and absorption, of the law of entropy, and of the proposition concerning the existence of roots of an equation.

‖8‖ But critical examination of these 'proofs' shows that they are not in themselves proofs, but basically only make it possible to trace things back to certain deeper propositions, which in turn are now to be regarded as new axioms instead of the propositions to be proved. The actual so-called *axioms* of geometry, arithmetic, statics, mechanics, radiation theory, or thermodynamics arose in this way. These axioms form a layer of axioms which lies deeper than the axiom-layer given by the recently-mentioned fundamental theorems of the individual field of knowledge. The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual domains of knowledge—a deepening that is necessary for every edifice that one wishes to expand and to build higher while preserving its stability.

‖9‖ If the theory of a field of knowledge—that is, the framework of concepts that represents it—is to serve its purpose of orienting and ordering, then it must satisfy two requirements above all: *first* it should give us an overview of the *independence* and *dependence* of the propositions of the theory; *second*, it should give us a guarantee of the *consistency* of all the propositions of the theory. In particular, the axioms of each theory are to be examined from these two points of view.

‖10‖ Let us first consider the independence or dependence of the axioms.

‖11‖ The *axiom of parallels* in geometry is the classical example of the independence of an axiom. When he placed the parallel postulate among the axioms, Euclid thereby denied that the proposition of parallels is implied by the other axioms. Euclid's method of investigation became the paradigm for axiomatic research, and since Euclid geometry has been the prime example of an axiomatic science.

‖12‖ Classical mechanics furnishes another example of an investigation of the independence of axioms. The Lagrangian equations of motion were temporarily able to count as axioms of mechanics—for mechanics can of course be entirely based on these equations when they are generally formulated for arbitrary forces and arbitrary side-constraints. But further investigation shows that it is not necessary in the construction of mechanics to presuppose arbitrary forces or arbitrary side-constraints; thus the system of presuppositions can be reduced. This piece of knowledge leads, on the one hand, to the axiom system of Boltzmann, who assumes only forces (and indeed special central forces) but no side-constraints, and the axiom system of Hertz, who discards forces and makes do with side-constraints (and indeed special side-constraints with rigid connections). These two axiom systems form a deeper layer in the progressive axiomatization of mechanics.

⟦13⟧ If in establishing the theory of Galois equations we assume as an axiom the existence of roots of an equation, then this is certainly a dependent axiom; for, as Gauss was the first to show, that existence theorem can be proved from the axioms of arithmetic.

⟦14⟧ Something similar would happen if we were to assume as an axiom in the theory of prime numbers the proposition about the reality of the zeroes of the Riemann $\zeta(s)$-function: as we progress to a deeper layer of purely arithmetical axioms the proof of this reality-proposition would become necessary, and only this proof would guarantee the reliability of the important conclusions which we have already achieved for the theory of prime numbers by taking it as a postulate.

⟦15⟧ A particularly interesting question for axiomatics concerns the independence of the propositions of a field of knowledge from the axiom of *continuity*.

⟦16⟧ In the theory of real numbers it is shown that the axiom of measurement—the so-called Archimedean axiom—is independent of all the other arithmetical axioms. As everybody knows, this information is of great significance for geometry; but it seems to me to be of capital interest for physics as well, for it leads to the following result: the fact that by adjoining terrestrial distances to one another we can achieve the dimensions and distances of bodies in outer space (that is, that we can measure heavenly distances with an earthly yardstick) and the fact that the distances within an atom can all be expressed in terms of metres—these facts are not at all a mere logical consequence of propositions about the congruence of triangles or about geometric configurations, but are a result of empirical research. The validity of the Archimedean axiom in nature stands in just as much need of confirmation by experiment as does the familiar proposition about the sum of the angles of a triangle.

⟦17⟧ In general, I should like to formulate the axiom of continuity in physics as follows: 'If for the validity of a proposition of physics we prescribe any degree of accuracy whatsoever, then it is possible to indicate small regions within which the presuppositions that have been made for the proposition may vary freely, without the deviation of the proposition exceeding the prescribed degree of accuracy.' This axiom basically does nothing more than express something that already lies in the essence of experiment; it is constantly presupposed by the physicists, although it has not previously been formulated.

⟦18⟧ For example, if one follows Planck and derives the second law of thermodynamics from the axiom of the impossibility of a *perpetuum mobile of the second sort*, then this axiom of continuity must be used in the derivation.

⟦19⟧ By invoking the theorem that the continuum can be well-ordered, Hamel has shown in a most interesting manner that, in the foundations of statics, the axiom of continuity is necessary for the proof of the theorem concerning the *parallelogram of forces*—at any rate, given the most obvious choice of other axioms.

⟦20⟧ The axioms of classical mechanics can be deepened if, using the axiom of continuity, one imagines continuous motion to be decomposed into small straight-line movements caused by discrete impulses and following one another in rapid succession. One then applies Bertrand's maximum principle as the essential axiom of mechanics, according to which the motion that actually occurs after each impulse is that which maximizes the kinetic energy of the system with respect to all motions that are compatible with the law of the conservation of energy.

⟦21⟧ The most recent ways of laying the foundations of physics—of electrodynamics in particular—are all theories of the continuum, and therefore raise the demand for continuity in the most extreme fashion. But I should prefer not to discuss them because the investigations are not yet completed.

⟦22⟧ We shall now examine the second of the two points of view mentioned above, namely, the question concerning the *consistency* of the axioms. This question is obviously of the greatest importance, for the presence of a contradiction in a theory manifestly threatens the contents of the entire theory.

⟦23⟧ Even for successful theories that have long been accepted, it is difficult to know that they are internally consistent: I remind you of the reversibility and recurrence paradox in the kinetic theory of gases.

⟦24⟧ It often happens that the internal consistency of a theory is regarded as obvious, while in reality the proof requires deep mathematical developments. For example, consider a problem from the elementary theory of the *conduction of heat*—namely, the distribution of temperatures within a homogeneous body whose surfaces are maintained at a definite temperature that varies from place to place: then in fact the requirement that there be an equilibrium of temperatures involves no internal theoretical contradiction. But to know this it is necessary to prove that the familiar boundary-value problem of potential theory is always solvable; for only this proof shows that a temperature distribution satisfying the equations of the conduction of heat is at all possible.

⟦25⟧ But particularly in physics it is not sufficient that the propositions of a theory be in harmony with each other; there remains the requirement that they not contradict the propositions of a neighbouring field of knowledge.

⟦26⟧ Thus, as I showed earlier, the axioms of the elementary theory of radiation can be used to prove not only *Kirchhoff's law* of emission and absorption, but also a special law about the reflection and refraction of individual beams of light, namely, the law: If two beams of natural light and of the same energy each fall on the surface separating two media from different sides in such a way that one beam after its reflection, and the other after its passage, each have the same direction, then the beam that arises from uniting the two is also of natural light and of the same energy. This theorem is, as the facts show, not at all in contradiction with optics, but can be derived as a conclusion from the electromagnetic theory of light.

⟦27⟧ As is well known, the results of the *kinetic theory of gases* are in full harmony with *thermodynamics*.

⟦28⟧ Similarly, *electrodynamic inertia* and *Einsteinian gravitation* are compatible with the corresponding concepts of the classical theories, since the classical concepts can be conceived as limiting cases of the more general concepts in the new theories.

[29] In contrast, *modern quantum theory* and our developing knowledge of the internal structure of the atom have led to laws which virtually contradict the earlier electrodynamics, which was essentially built on the Maxwell equations; modern electrodynamics therefore needs—as everybody acknowledges—a new foundation and essential reformulation.

[30] As one can see from what has already been said, the contradictions that arise in physical theories are always eliminated by changing the selection of the axioms; the difficulty is to make the selection so that all the observed physical laws are logical consequences of the chosen axioms.

[31] But matters are different when contradictions appear in purely theoretical fields of knowledge. Set theory contains the classic example of such an occurrence, namely, in the *paradox of the set of all sets*, which goes back to Cantor. This paradox is so serious that distinguished mathematicians, for example, Kronecker and Poincaré, felt compelled by it to deny that set theory—one of the most fruitful and powerful branches of knowledge anywhere in mathematics—has any justification for existing.

[32] But in this precarious state of affairs as well, the axiomatic method came to the rescue. By setting up appropriate axioms which in a precise way restricted both the arbitrariness of the definitions of sets and the admissibility of statements about their elements, Zermelo succeeded in developing set theory in such a way that the contradictions disappear, but the scope and applicability of set theory remain the same.

[33] In all previous cases it was a matter of contradictions that had emerged in the course of the development of a theory and that needed to be eliminated by a reformulation of the axiom system. But if we wish to restore the reputation of mathematics as the exemplar of the most rigorous science it is not enough merely to avoid the existing contradictions. The chief requirement of the theory of axioms must go farther, namely, to show that within every field of knowledge contradictions based on the underlying axiom-system are *absolutely impossible*.

[34] In accordance with this requirement I have proved the consistency of the axioms laid down in the *Grundlagen der Geometrie* by showing that any contradiction in the consequences of the geometrical axioms must necessarily apppear in the arithmetic of the system of real numbers as well.

[35] For the fields of physical knowledge too, it is clearly sufficient to reduce the problem of *internal consistency* to the consistency of the arithmetical axioms. Thus I showed the consistency of the axioms of the *elementary theory of radiation* by constructing its axiom system out of analytically independent pieces—presupposing in the process the consistency of analysis.

[36] One may and should in some circumstances proceed similarly in the construction of a mathematical theory. For example, if in the development of the theory of Galois groups we have taken the proposition of the *existence of roots* as an axiom, or if in the theory of prime numbers we have taken the

hypothesis concerning the *reality of the zeros* of the Riemann $\zeta(s)$-function as an axiom, then in each case the proof of the consistency of the axiom system comes down to a proof, using the means of analysis, of the proposition of the existence of roots or of the Riemann hypothesis concerning $\zeta(s)$—and only then has the theory been securely completed.

⟦37⟧ The problem of the consistency of the axiom system for the *real numbers* can likewise be reduced by the use of set-theoretic concepts to the same problem for the integers: this is the merit of the theories of the irrational numbers developed by Weierstrass and Dedekind.

⟦38⟧ In only two cases is this method of reduction to another special domain of knowledge clearly not available, namely, when it is a matter of the axioms for the *integers* themselves, and when it is a matter of the foundation of *set theory*; for here there is no other discipline besides logic which it would be possible to invoke.

⟦39⟧ But since the examination of consistency is a task that cannot be avoided, it appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic.

⟦40⟧ This method was prepared long ago (not least by Frege's profound investigations); it has been most successfully explained by the acute mathematician and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the *axiomatization of logic* as the crowning achievement of the work of axiomatization as a whole.

⟦41⟧ But this completion will require further work. When we consider the matter more closely we soon recognize that the question of the consistency of the integers and of sets is not one that stands alone, but that it belongs to a vast domain of difficult epistemological questions which have a specifically mathematical tint: for example (to characterize this domain of questions briefly) the problem of the *solvability in principle of every mathematical question*, the problem of the subsequent *checkability* of the results of a mathematical investigation, the question of a *criterion of simplicity* for mathematical proofs, the question of the relationship between *content and formalism* in mathematics and logic, and finally the problem of the *decidability* of a mathematical question in a finite number of operations.

⟦42⟧ We cannot rest content with the axiomatization of logic until all questions of this sort and their interconnections have been understood and cleared up.

⟦43⟧ Among the mentioned questions, the last—namely, the one concerning decidability in a finite number of operations—is the best-known and the most discussed; for it goes to the essence of mathematical thought.

⟦44⟧ I should like to increase the interest in this question by indicating several particular mathematical problems in which it plays a role.

⟦45⟧ In the theory of *algebraic invariants* we have the fundamental theorem that there is always a finite number of whole rational invariants by means of which all other such invariants can be represented. In my opinion, my first general proof of this theorem completely satisfied our requirements of simplicity and perspicuity; but it is impossible to reformulate this proof so that we can obtain from it a statable bound for the number of the finitely many invariants of the full system, let alone obtain an actual listing of them. Instead, new principles and considerations of a completely different sort were necessary in order to show that the construction of the full system of invariants requires only a finite number of operations, and that this number is less than a bound that can be stated before the calculation.

[46] We see the same thing happening in an example from the *theory of surfaces*. It is a fundamental question in the geometry of surfaces of the fourth order to determine the maximum number of separate sheets it takes to make up such a surface.

[47] The first step towards an answer to this question is the proof that the number of sheets of a curved surface must be finite. This can easily be shown function-theoretically as follows. One assumes the existence of infinitely many sheets, and selects a point inside each spatial region bounded by a sheet. A point of accumulation for these infinitely many chosen points would then be a point of a singularity that is excluded for an algebraic surface.

[48] This function-theoretic path does not at all lead to an upper bound for the number of surface-sheets. For that, we need instead certain observations on the number of cut-points, which then show that the number of sheets certainly cannot be greater than 12.

[49] The second method, entirely different from the first, in turn cannot be applied or transformed to decide whether a surface of the fourth order with twelve sheets actually exists.

[50] Since a quaternary form of the fourth order possesses 35 homogeneous coefficients, we can conceive of a given surface of the fourth order as a point in 34-dimensional space. The discriminant of the quaternary form of fourth order is of degree 108 in its coefficients; if it is set equal to zero, it accordingly represents in 34-dimensional space a surface of order 108. Since the coefficients of the discriminant are themselves determinate integers, the topological character of the discriminant surface can be precisely determined by the rules that are familiar to us from 2- and 3-dimensional space; so we can obtain precise information about the nature and significance of the individual subdomains into which the discriminant surface partitions the 34-dimensional space. Now, the surfaces of fourth order represented by points of these subdomains all certainly possess the same sheet-number; and it is accordingly possible to establish, by a long and wearying but finite calculation, whether we have a surface of fourth order with $n \leq 12$ sheets or not.

[51] The geometric method just described is thus a third way of treating our question about the maximum number of sheets of a surface of the fourth order. It proves the decidability of this question in a finite number of operations. So in principle an important demand of our problem has been satisfied: it has been reduced to a problem of the level of difficulty of determining the $10^{(10^{10})}$th numeral in the decimal expansion for π—a task which is clearly solvable, but which remains unsolved.

⟦52⟧ Rather, it took a profound and difficult algebraic-geometric investigation by Rohn to show that 11 sheets are not possible in a surface of the fourth order, while 10 sheets actually occur. Only this fourth method delivered the full solution of the problem.

⟦53⟧ These particular discussions show how a variety of methods of proof can be applied to the same problem, and they ought to suggest how necessary it is to study the essence of mathematical proof itself if one wishes to answer such questions as the one about decidability in a finite number of operations.

⟦54⟧ All such questions of principle, which I characterized above and of which the question just discussed—that is, the question about decidability in a finite number of operations—was only the last, seem to me to form an important new field of research which remains to be developed. To conquer this field we must, I am persuaded, make the concept of specifically mathematical proof itself into an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

⟦55⟧ To be sure, the execution of this programme is at present still an unsolved task.

⟦56⟧ In conclusion, I should like to sum up in a few sentences my general conception of the essence of the axiomatic method. I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms in the sense explained above we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science.

The German text of Hilbert's *Axiomatisches Denken* is from the *Mathematische Annalen*, vol. 78, pp. 405-415, 1918.

The English translation *Axiomatic thought* is by William Ewald from *William Ewald (ed.): From Kant to Hilbert*, volume II, Oxford University Press, pp.1105-1115.