Type theories, (intuitionistic) set theories and univalence

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Axiomatic Thinking

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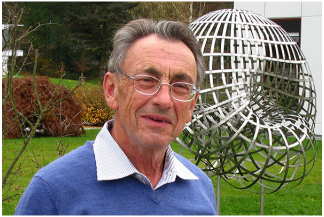
Hilbert 1930

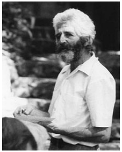
,,Wir müssen wissen! Wir werden wissen!"

Dedicated to the memory of

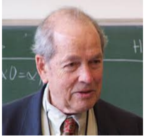
- Grigori Mints (June 7, 1939 - May 29, 2014)



- William Craig (November 13, 1918 - January 13, 2016)



- Solomon Feferman (December 13, 1928 - July 26, 2016)



- Jack Silver (April 23, 1942 - December 22, 2016)

- Gaisi Takeuti (January 25, 1926 - May 10, 2017)

- Vladimir Voevodsky (June 4, 1966 - September 30, 2017)

Why Intuitionistic Theories?

- Philosophical Reasons: **Brouwer, Dummett, Martin-Löf, Feferman, Linnebo, ...**
- Computational content: Witness and program extraction from proofs.
- Intuitionistically proved theorems hold in more generality: The internal logic of most topoi is intuitionistic logic.
- Axiomatic Freedom Adopt axioms that are classically refutable but interesting and desirable.

Axiomatic Freedom or “New Worlds”

- May be it would be nice
- if all $f : \mathbb{N} \to \mathbb{N}$ were computable and those pesky non-standard models of **PA** didn’t exist?
- if all $f : \mathbb{R} \to \mathbb{R}$ were continuous and the world were Brouwerian?
- if all functions between manifolds were differentiable? (nilpotent non-zero infinitesimals)
- if there existed a set A with $\mathbb{N} \subseteq A$ such that A is in 1-1 correspondence with $A \to A$?
- if all $f : \mathbb{R} \to \mathbb{R}$ were measurable?
- if all homotopically equivalent sets could be viewed as identical (univalence)?

Recent popularity of type theory

Scientific American, Quanta Magazine, Nautilus, ...

> *Voevodsky's Univalent Foundations require not just one inaccessible cardinal but an infinite string of cardinals, each inaccessible from its predecessor.*

Michael Harris, *Mathematics without apologies*, 2015.

Ian Hacking, *Why is there Philosophy of Mathematics at All?*, 2014.

Some "research" questions

- Take Martin-Löf type theory with all type constructors (**MLTT**), including W-types and infinitely many universes

$$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$$

- How strong is this theory?
- Not difficult to show that **ZFC** plus infinitely many inaccessibles is an upper bound.
- How strong is **MLTT** plus univalence for all universes?
- Now add the impredicative type **Prop** of propositions together with

$$\text{Prop} : \mathcal{U}_0$$

  How strong is this theory? (aka Calculus of inductive Constructions (CiC)).
- What are the set-theoretic counterparts (intuitionistic set theories) of such type theories?

Type theory

- Types are structured collections of objects such as natural numbers.
- 1908 Russell:
  Mathematical logic as based on the theory of types
- 1910, 1912, 1913 Russell & Whitehead:
  Principia Mathematica
- 1926 Hilbert: *Über das Unendliche*
- 1940 Church: *A formulation of the simple theory of types*
- 1967 de Bruijn: *AUTOMATH*
- 1971 Martin-Löf: *A Theory of Types*

MLTT Judgements

A **judgement** has one of the following four forms:

- A type
  ("A is a well-formed type")
- $A = B$ type
  ("A and B are equal well-formed types")
- $a : A$
  ("a is a well-formed term of type A")
- $a = b : A$
  ("a and b are equal well-formed terms of type A")

Martin-Löf type theory as a deductive systems

One deduces sequents

$$\Gamma \vdash \mathfrak{A}$$

where Γ, called the **context**, is made up of variable declarations $(x : A)$ in the "right" order of dependency, and $\mathfrak{A}$ is a judgement.

The rules are divided into formation, introduction, elimination and equality rules.

The basic dependent type theory $\mathbf{MLTT}_{basic}$

$\mathbf{MLTT}_{basic}$ is the dependent type theory with the following forms of type:

- Bool, Empty and the type Nat of natural numbers.
- $\mathrm{List}(A)$, $A + B$ and $\mathrm{Id}(A, a, b)$.
- Dependent product: $\prod_{x:A} B(x)$
- Dependent sum: $\sum_{x:A} B(x)$

The Curry-Howard representation of the logical operations

- The standard approach of representing logic in Martin-Löf type theory is to view propositions (formulae, sentences) as types.
- The Σ type represents $\exists$.
- The Π type represents $\forall$.
- The $\times$ type represents $\wedge$.
- The $+$ type represents $\vee$.
- The $\rightarrow$ type represents $\supset$.
- Empty represents falsum.
- $\mathrm{Id}(A, a, b)$ to represent equality on A.

The full system **MLTT**

- has the W-type

$$\mathrm{W}_{x:A} B(x)$$

which is the type of *well-founded trees* over the family of types $(B(x))_{x:A}$.

W-types are a generalization of such types as natural numbers, lists, binary trees. They capture the "recursion" aspect of any inductive type.

- And it has infinitely many universes

$$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$$

- A universes is a type inhabited by types. Every universe is closed under all the previous type constructions and $\mathcal{U}_i : \mathcal{U}_{i+1}$.

Universes and Notation

- Universes $\mathcal{U}$ are types that contain types as elements.
- They contain Bool, Empty, Nat, and are closed under all the (other) type forming operations. E.g.

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash (\prod_{x:A} B(x) : \mathcal{U}}$$

- Denote by $\mathbf{MLTT}^{-}$ the theory $\mathbf{MLTT}$ without W-types.
- $\mathbf{MLTT}_n$ is the subsystem with only n universes $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$.

  Furthermore, $\mathbf{MLTT}_n^{-}$ also lacks the W-type constructor.

Two Identities

- General equality rules (reflexivity, symmetry, transitivity) and substitution rules, simultaneously at the level of terms and types, apply to judgements. Re-write rules.

- But there is also propositional identity which gives rise to types $\mathsf{Id}(A, s, t)$ and allows for internal reasoning about identity.

  Shall write $s =_A t$ rather than $\mathsf{Id}(A, s, t)$

Higher identity structure on any type A

$$a =_A a'$$

$$p =_{a =_A a'} p'$$

$$\theta =_{p =_{a =_A a'} p'} \theta'$$

$$\vdots$$

In extensional type theory (Martin-Löf 1979, 1984) this hierarchy collapses, since $a =_A a'$ contains at most 1 element.

Not so in intensional type theory (Martin-Löf 1973, 1986). Groupoid model (Hofmann, Streicher 1994), Kan simplicial sets (Voevodsky 2010), Kan cubical sets (Bezem, Coquand, Huber 2013).

Extensional identity

(Id–Formation) $$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a =_A b \text{ type}}$$

(Id–Introduction) $$\frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : a =_A a}$$

(Id–Uniqueness) $$\frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash p = 1_a : a =_A b}$$

(Id–Reflection) $$\frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash a = b : A}.$$

- Reflection makes judgemental identity undecidable, i.e., the (type checking) questions whether $\Gamma \vdash a = b : A$ or $\Gamma \vdash a : A$ hold become undecidable.

New identity laws, Martin-Löf 1973

Indiscernability of Identicals:

If $p : a =_A b$ and $P(a)$ then $P(b)$.

This entails a transport function $t(p) : P(a) \to P(b)$.

Generalization: Now suppose that

$$d(x) : C(x, x, 1_x)$$

holds for all $x : A$.

Then d can be extended to a function $\tilde{J}_d$ on

$$\sum_{x,y : A} x =_A y$$

i.e., if $a, b : A$ and $p : a =_A b$ then

$$\begin{aligned} \tilde{J}_d(a, b, p) &\quad : \quad C(a, b, p) \\ d(a) = \tilde{J}_d(a, a, 1_a) &\quad : \quad C(a, a, 1_a) \end{aligned}$$

Rules for intensional identity

$$
\text{(Id–Elim)} \quad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \\ \Gamma \vdash p : a =_A b \\ \Gamma, x : A, y : A, z : x =_A y \vdash C(x, y, z) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathrm{J}(d, a, b, p) : C(a, b, p)}
$$

$$
\text{(Id–Eq)} \quad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma, x : A, y : A, p : x =_A y \vdash C(x, y, p) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathrm{J}(d, a, a, 1_a) = d(a) : C(a, a, 1_a)}
$$

Strengths of **MLTT**?

- 1980s work on Martin-Löf type theory by Aczel, Beeson, Feferman, Hancock, Jervell, ...
- Early 1990's: proof-theoretic tools were in place to determine the exact strength of Martin-Löf type theories with finitely many universes, infinitely many universes, W-types, no W-types, super univ., Mahlo-univ., etc.
- E. Palmgren (1992)
- R. (1993)
- A. Setzer (1998)

Myhill's Constructive set theory 1975

CST based on intuitionistic logic

Many sorted system: **numbers, sets, functions**

Axioms (simplified)

- **Extensionality**
- **Pairing**, **Union**, **Infinity** (or $\mathbb{N}$ is a set)
- **Bounded Separation**
- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.
- **Replacement**
- **Set Induction Scheme**

Moving between type theory and set theory

The types-as-sets interpretation (TaS).

$$\text{type theory} \hookrightarrow \text{set theory}$$

Aczel (late 1970's): The sets-as-trees interpretation (SaT)

$$\text{set theory} \hookrightarrow \text{type theory}$$

- R., S. Tupailo, Characterizing the interpretation of set theory in Martin-Löf type theory, APAL 2006.
- Cesare Galozzi, Variations: Uses *h*-sets as index sets for the interpretation.

Constructive Zermelo-Fraenkel set theory, **CZF**

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Bounded Separation**
- **Subset Collection**

  For all sets A, B there exists a "sufficiently large" set of multi-valued functions from A to B.

- **Strong Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \ \rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)\ \wedge\ (\forall y \in b)\, (\exists x \in a)\ \varphi(x, y)\,]$$

- **Set Induction scheme**

Three notions of large set

- A set A is said to be **regular** if it is inhabited and transitive and whenever $B \in A$ and R is a set relation such that $\forall x \in B\, \exists y \in A\; R(x, y)$ then there exists $C \in A$ such that $\forall x \in B\, \exists y \in C\; R(x, y)$ and $\forall y \in C\, \exists x \in B\; R(x, y)$.
- Denote by $\mathbf{CZF}^-$ the theory $\mathbf{CZF}$ without the Set Induction scheme.
- A set I is said to be **weakly inaccessible** if I is a regular set such that $I \models \mathbf{CZF}^-$.
- A set I will be called **inaccessible** if I is weakly inaccessible and for all $x \in I$ there exists a regular set $y \in I$ such that $x \in y$.

An ‘algebraic’ characterization of “inaccessibility”

Proposition ($\mathbf{CZF}^-$)

A set I is weakly inaccessible iff I is a regular set such that the following are satisfied:

1. $\omega \in I$,
2. $\forall a \in I \ \bigcup a \in I$,
3. $\forall a \in I \, [a \text{ inhabited } \Rightarrow \bigcap a \in I]$,
4. $\forall A, B \in I \, \exists C \in I \quad C$ is full in $\mathbf{mv}({}^{A}B)$.

How strong is $\mathbf{MLTT}^-$ plus Univalence?

Recall that $\mathbf{CZF}^-$ denotes the theory $\mathbf{CZF}$ without the Set Induction scheme.

Theorem 1. (Crosilla, R. 2002)

The theory

$$\mathbf{CZF}^- + \forall x\, \exists I\, [x \in I \ \wedge \ I \text{ weakly inaccessible}]$$

has the same strength as

$$\mathbf{ATR}_0$$

so has proof-theoretic ordinal Γ_0.

Proposition. $\mathbf{MLTT}^-$ can be interpreted in

$$\mathbf{CZF} + \text{weak-INACC}$$

where weak-INACC stands for $\forall x\, \exists I\, [x \in I \ \wedge \ I \text{ weakly inaccessible}]$.

Theorem 2. $\mathbf{MLTT}^- + \mathrm{UA}$ can be interpreted in $\mathbf{CZF} + \text{weak-INACC}$, too.

Here UA asserts that all universes are univalent.

The Bezem-Coquand-Huber constructive Kan cubical sets model can be done in this theory.

Corollary. All the theories $\mathbf{MLTT}^-$, $\mathbf{CZF} + \text{weak-INACC}$, and $\mathbf{MLTT}^- + \mathbf{UA}$ are of the same strength.

It does not matter whether the identity type is extensional or intensional.

It was known by work of Jervell 1978 and Feferman 1980 that (extensional) $\mathbf{MLTT}^-$ has strength Γ_0.

Univalence

- Let $f, g : \prod_{x:A} P(x)$. A **homotopy** from f to g is a dependent function of type
$$(f \simeq g) :\equiv \prod_{x:A} (f(x) =_{P(x)} g(x)).$$
- Let $f : A \vdash B$.
$$\mathrm{isequiv}(f) :\equiv \Big(\sum_{g:B\to A} (f \circ g \simeq \mathrm{id}_B)\Big) \times \Big(\sum_{h:B\to A} (h \circ f \simeq \mathrm{id}_A)\Big).$$
- $(A \simeq B) :\equiv \sum_{f:A\to B} \mathrm{isequiv}(f)$.
- For types $A, B : \mathcal{U}$ there is a canonical function
$$\mathrm{idtoeqv} : (A =_{\mathcal{U}} B) \vdash (A \simeq B).$$
The **Univalence Axiom** asserts that this function is itself an equivalence:
$$(A =_{\mathcal{U}} B) \simeq (A \simeq B).$$

Strength of **MLTT**

Theorem:

The following theories prove the same arithmetical statements:

(i) **MLTT**.

(ii) *The extensional type theory* $\mathbf{MLTT}^{ext}$.

(iii) **CZF** *plus for every* $n \in \mathbb{N}$, *an axiom asserting that there is a tower of n-many inaccessible sets,* $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n$.

(iv) $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n + \mathrm{RDC} +$ *Presentation Ax,*

where RDC *signifies the* relativized dependent choices axiom.

"Classical" Strength of **MLTT**

- It's the same as

$$\mathbf{KP} + \{n\text{-many recursively inaccessible ordinals}\}_{n \in \mathbb{N}}$$

or

$$\Delta^1_2\text{-CA} + \{n \text{ tower of } \beta\text{-models of } \Delta^1_2\text{-CA}\}_{n \in \mathbb{N}}$$

- The strength of all of these theories is considerable but tiny when compared to Π^1_2-$\mathbf{CA}_0$.
- Does the addition of the Univalence Axiom change that picture?
- No, since the cubical model of **Bezem, Coquand, Huber** can be done "constructively" in type theory, though not all types have been included yet.

For details see: M. Rathjen, *Proof Theory of Constructive Systems: Inductive Types and Univalence*, arXiv:1610.02191 (2016).

To appear in: *Feferman on Foundations: Logic, Mathematics, Philosophy*, G. Jäger, W. Sieg(Eds.), (Springer 2017).