Proofs as objects:

A pivotal thought

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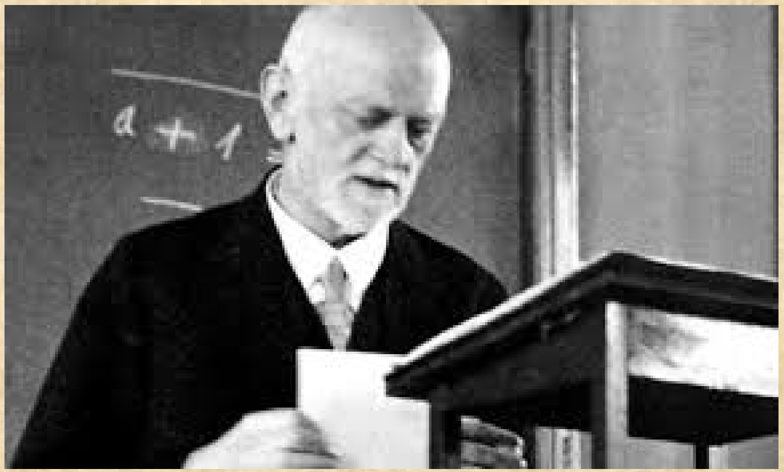
DAVID HILBERT (1862–1943)

THE GERMAN mathematician, physicist, and philosopher David Hilbert had enormous influence on mathematics at the beginning of the 20th century. In geometry, his influence has been likened to that of the ancient Greek mathematician Euclid. His work on axiomatic principles in this field was particularly significant. At the International Mathematical Congress in 1900, in Paris, France, he presented to the delegates 23 unsolved mathematical problems. These became known as Hilbert's problems, and although some have since been solved, others continue to challenge mathematicians. He is also remembered for work on logic and for his later research in physics.



Milestones

Axioms for geometry and analysis (1893-1903)

Syntactic consistency proof for a very small fragment of number theory (1904)

Regular lectures on foundations from 1905 -1917

Lectures *Prinzipien der Mathematik* (1917-18)

Lectures and papers on the finitist consistency program (1921-1931)

Hilbert, Zürich 1917

We must – that is my conviction – take the concept of the specifically mathematical proof as an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

This is a group portrait of the Swiss Mathematical Society in Zürich in 1917. I cannot identify all of these at this point, but you will see here some rather well-known people in the front row: Constantin Carathéodory, Marcel Grossmann, David Hilbert, K. F. Geiser, Hermann Weyl and his wife. And towards the back, Ferdinand Gonseth, Andreas Speiser, Michel Plancherel, Erich Hecke, Paul Bernays, and Otto Spiess. It was taken in front of the Landesmuseum in Zürich.
This is the Grossmann with whom Einstein wrote his first paper on general relativity. After that paper Grossmann said: "My main merit about this paper is that I did not become crazy."

Overview

Part I: Structural axiomatics and consistency

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Part II: Formal axiomatics and of proof theory

Hilbert, Hamburg1927

The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.

Overview

Part I: Structural axiomatics and consistency

Part II: Formal axiomatics and proof theory

Remarks: The 24^{th} problem and proof search

PART I

Structural axiomatics and consistency

Hilbert's Second Problem

… I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: *To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results*.

The setting for geometry

Here [in geometry] one begins customarily by assuming the existence of all the elements, i.e. one postulates at the outset three systems of things (namely, the points, lines, and planes), and then – essentially after the model of Euclid – brings these elements into relationship with one another by means of certain axioms of linking, order, congruence, and continuity.

Hilbert's structural definition I

We think three different systems of things: we call the things of the first system *points* …; we call the things of the second system *lines* …; we call the things of the third system *planes* …; We think the points, lines, planes in certain mutual relations ..; the precise and complete description of these relations is obtained by the *axioms of geometry*.

Hilbert's structural definition II

We think a system of things, and we call them [real] numbers and denote them by a, b, c … We think these numbers to be in certain mutual relations, whose precise and complete description is obtained through the following axioms.

Systems & things

What are the necessary and sufficient and mutually independent conditions a system of things has to be subjected to, so that to each property of these things a geometric fact corresponds, and conversely, thereby having these things provide a complete “image” of geometric reality.

Simply infinite system

A system N is *simply infinite* if and only if there is an element 1 and a mapping ϕ, such that the characteristic conditions $(\alpha) - (\delta)$ hold for them

Dedekind #71, 1888

Simply infinite system

Without a logical proof of existence, it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such a proof (articles 66 and 72 of my essay).

Letter to Keferstein

Direct consistency proof

I am convinced that it must be possible to find a direct proof for the consistency of the arithmetic axioms [as proposed in *Über den Zahlbegriff* for the real numbers], by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers.

Two problems

The necessary task then arises of showing the consistency and the completeness of these axioms, i.e., it must be proved that the application of the given axioms can never lead to contradictions, and, further, that the system of axioms suffices to prove all geometric [and arithmetic] propositions.

… Nowhere is there a statement of the logical or other laws on which he [Dedekind] builds, and, even if there were, we could not possibly find out whether really no others were used – for to make that possible the proof must be not merely indicated but completely carried out.

Frege

PART II

Formal axiomatics and proof theory

Hilbert, Zürich 1917

But since the examination of consistency is a task that cannot be avoided, it appears necessary to axiomatize logic itself and *to prove that number theory and set theory are only parts of logic*.

Hilbert & Bernays, 1918

Thus, it is clear that the introduction of the axiom of reducibility is the appropriate means to turn the ramified calculus into a system out of which the foundations of higher mathematics can be developed.

Hilbert & Bernays, 1921-22

The argument is sketched and the methodological approach is described in Hilbert's Leipzig talk of 1922. It was hoped that it could be extended quickly to full number theory and analysis!

Hilbert & Bernays, 1921-22

Axioms for logical connectives:

(A&B) -> A

(A&B) -> B

(A -> (B -> (A&B))

Similarly for disjunction etc.

Formalization and Proofs

The words of ordinary language are replaced by particular *signs*, the logical inference steps [are replaced by] rules that form new formally presented statements from already proved ones.

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The objects of proof theory shall be the *proofs* carried out in mathematics proper.

Gentzen 1936

Natural Formalization

Natural Formalization

- ***Intercalation***: bidirectional reasoning or reasoning with gaps.
- ***Definitional expansions***: Introduction and elimination rules for defined notions.
- ***Conceptual organization***: lemmas as rules.

Cantor-Bernstein Theorem

1.	f∈inj(a,b)	Prem
2.	g∈inj(b,a)	Prem
3.	1(g ∘ f)∈bij(a,g ∘ f[a])	Theorem (Core12) 1, 2
4.	g[b]⊆a	Theorem (Func17) 2
5.	g ∘ f[a]⊆g[b]	Theorem (Comp11) 1, 2
6.	a≈g[b]	Theorem (Fundamental Lemma) 3, 4, 5
7.	b≈g[b]	Theorem (Equi4) 2
8.	a≈b	Theorem (Equi8) 6, 7

Dedekind, 1887

Fundamental Lemma. Let h be a bijection from a to e and let d be a set with $e \subseteq d \subseteq a$; then there is a bijection h^* from a to d.

Problem 24

Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs.

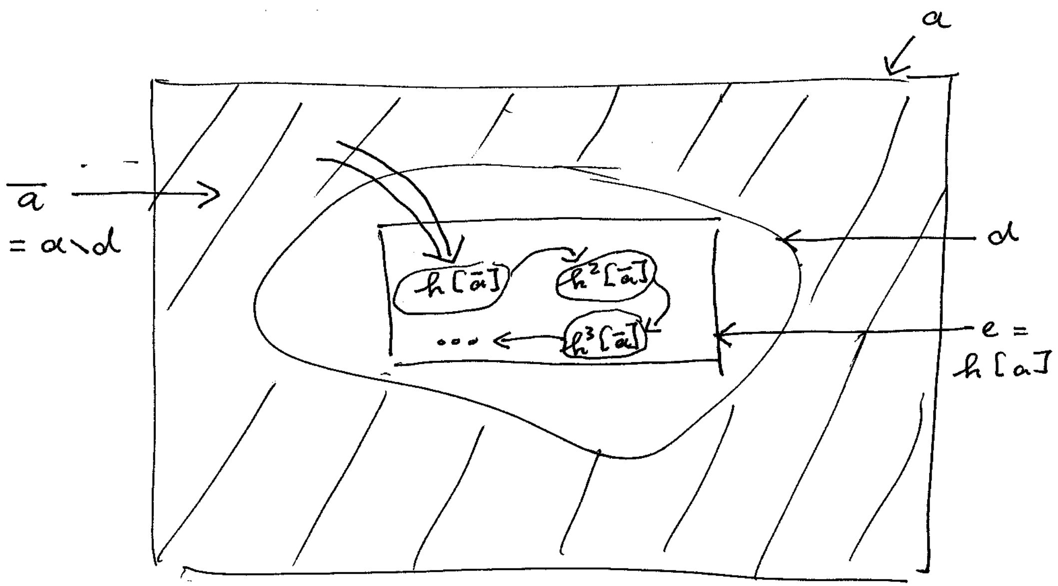


Figure 1

$h^0[\bar{a}] = \bar{a}$ and $h^{n+1}[\bar{a}] = h[h^n[\bar{a}]]$

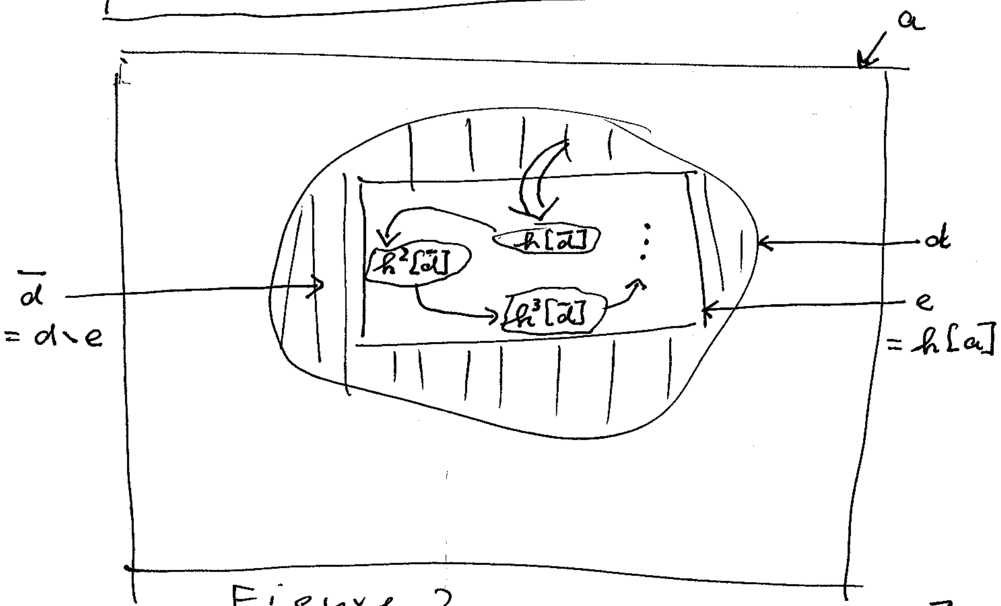


Figure 2

Proof analysis

There is essentially ONE proof, Dedekind's, and one variation, Zermelo's. In addition, the bijections obtained from the various proofs are either Dedekind's or Zermelo's. And, finally, at the heart of the considerations is one *central fact* for inductively defined sets!

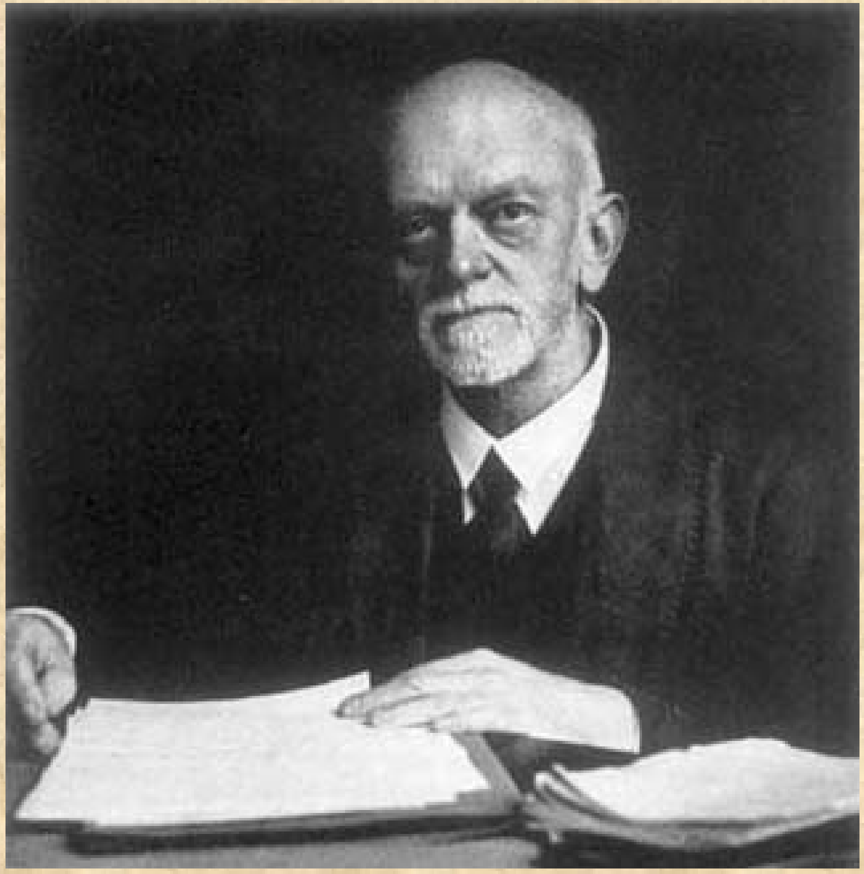
Turing 1951

The whole thinking process is still rather mysterious to us, but I believe that the attempt to make a thinking machine will help us greatly in finding out how we think ourselves.

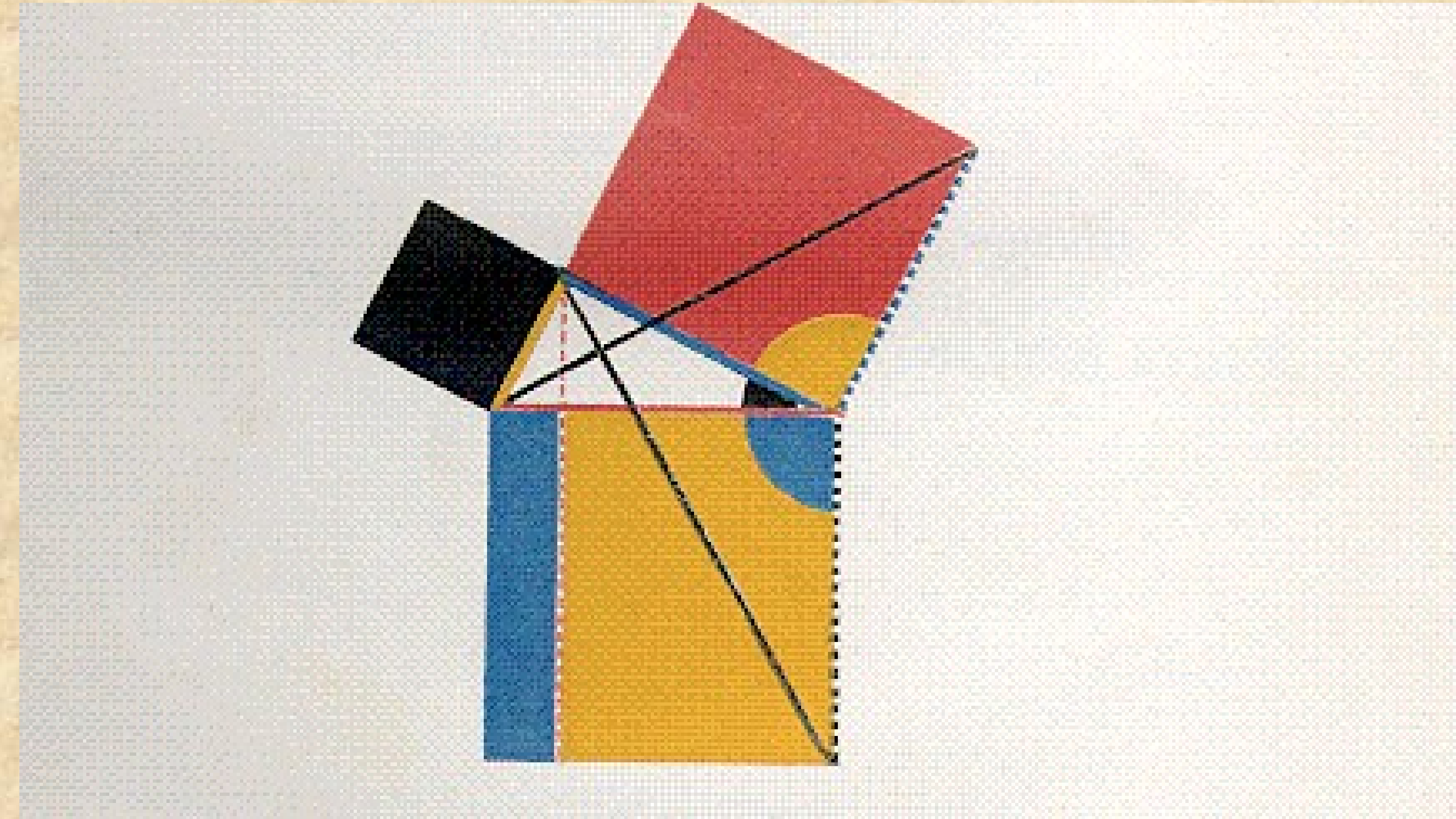
Decidability

The last of the listed questions, namely, the question concerning the decidability by means of finitely many operations, is the best-known and most frequently discussed one *because it deeply touches the essence of mathematical thinking*.

Hilbert 1932



Byrne's diagram



Elementary geometric facts

- SAS
- Triangles are equal when they have the same base and when their third vertex lies on the same parallel to the base.
- A diagonal divides a rectangle into two equal triangles.

Byrne (with labels)