Foundations of mathematics: an optimistic message

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Foundations of mathematics: an optimistic message

Speaker: Stephen G. Simpson
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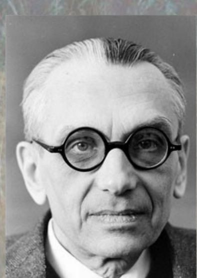
Venue: LT31, Block S16, Level 3
Faculty of Science
National University of Singapore
10 Lower Kent Ridge Road Singapore 117546

Plato and Aristotle

"The infinite! No other question has ever moved so profoundly the spirit of man." -- David Hilbert (1862-1943)

About the Speaker

Stephen G. Simpson is a senior mathematician and mathematical logician. He is prominent as a researcher in the foundations of mathematics. His writings have been influential in promoting the foundations of mathematics as an exciting research area.



"Objective concepts of mathematics are fundamental to my work in logic."
-- Kurt Gödel (1906-1978)

Abstract

Historically, mathematics has been regarded as a role model for all of science -- a paragon of abstraction, logical precision, and objectivity. The 19th and early 20th centuries saw tremendous progress. The great mathematician David Hilbert proposed a sweeping program whereby the entire panorama of higher mathematical abstractions would be justified objectively and logically, in terms of finite processes. But then in 1931 the great logician Kurt Gödel published his famous incompleteness theorems, thus initiating an era of confusion and skepticism. In this talk I show how modern foundational research has opened a new path toward objectivity and optimism in mathematics.

Free Admission

For more information, visit www2.ims.nus.edu.sg and www.math.psu.edu/simpson/talks/nus1601/.

Kurt Gödel. 17 November 2015. Online image. Retrieved from http://guncelmatematik.com/kurt-godel-kimdir.html
David Hilbert. 17 November 2015. Online image. Retrieved from http://davidhilbertmth482.blogspot.in/
Plato and Aristotle. 24 November 2015. Online image. Retrieved from https://en.wikipedia.org/wiki/Philosophy#/media/File:Sanzio_01_Plato_Aristotle.jpg

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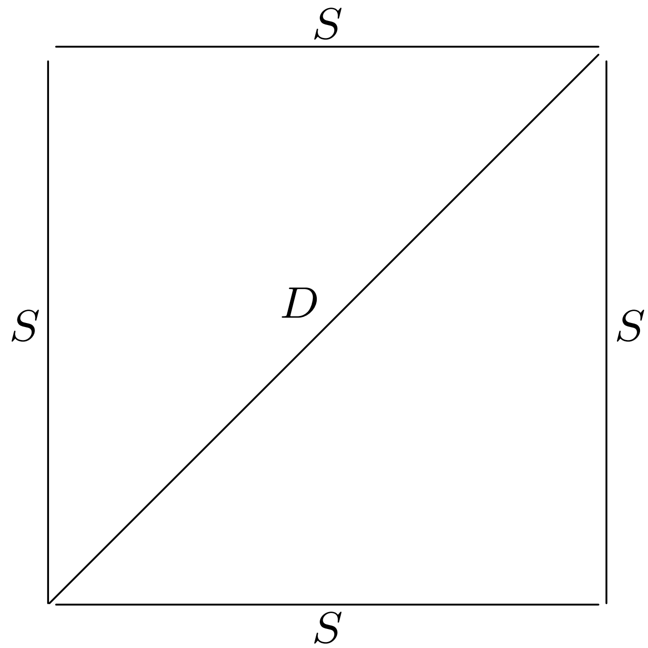
The School of Athens (about 360 B.C.)

by Raphael (1483–1520)



$$\sqrt{2} = \text{the square root of } 2 = \frac{D}{S}.$$

$$\sqrt{2} \text{ is } \underline{\text{approximately}} \text{ equal to } \frac{99}{70}.$$

$$\sqrt{2} \text{ is } \underline{\text{approximately}} \text{ equal to } \frac{665857}{470832}.$$

$$\sqrt{2} \text{ is ``}\underline{\text{exactly}}\text{'' equal to } 1.4142135623730950488016\ldots.$$

Plato and Aristotle (about 360 B.C.)

David Hilbert (1862–1943)

"The infinite! No other question has ever moved so profoundly the spirit of man."

Kurt Gödel (1906–1978)

"Objective concepts of mathematics are fundamental to my work in logic."

Hilbert's Program (1926):

Using the tools of mathematical logic
Hilbert proposed to prove that <u>all</u> of mathematics,
including the infinitistic parts of mathematics,
is <u>reducible</u> to <u>purely finitistic</u> mathematics.

In this way, the objectivity of mathematics
would be confirmed.

Gödel's refutation of Hilbert's Program (1931):

Gödel used mathematical logic to prove that
<u>some parts</u> of infinitistic mathematics
<u>are not</u> reducible to finitistic mathematics.

This includes the "medium" and "strong" levels
of the Gödel hierarchy.

Thus, the objectivity of mathematics is left in doubt.

The Gödel hierarchy

“strong”

$\vdots$
huge cardinal numbers
$\vdots$
ineffable cardinal numbers
$\vdots$
ZFC (Zermelo/Fraenkel set theory)
ZC (Zermelo set theory)
simple type theory

“medium”

Z_2 (second-order arithmetic)
$\vdots$
Π^1_2-CA_0 (Π^1_2 comprehension)
Π^1_1-CA_0 (Π^1_1 comprehension)
ATR_0 (arithmetical transfinite recursion)
ACA_0 (arithmetical comprehension)

“weak”

WKL_0 (weak König’s lemma)
RCA_0 (recursive comprehension)
PRA (primitive recursive arithmetic)
EFA (elementary function arithmetic)
bounded arithmetic
$\vdots$

Reverse Mathematics:

A series of precise case studies
to determine which parts of mathematics
belong to which levels of the Gödel hierarchy.

Two discoveries:

1. Using the methods envisioned by Hilbert, we can prove that "weak" levels of the Gödel hierarchy are finitistically reducible, in the sense of Hilbert's program.

2. Reverse-mathematical case studies provide solid evidence that the "weak" levels cover at least 85 percent of mathematics.

   This includes most or all of the "applicable" parts of mathematics.

Combining these two discoveries, we conclude that Hilbert's program is largely valid.

My optimistic message:

Most of mathematics has an objective basis!