A semilinear elliptic problem
with a singularity at \( u = 0 \)

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In this joint work with Daniela Giachetti (Rome, Italy) and Pedro J. Martínez Aparicio (Cartagena, Spain) (see [3] and [4]), we consider the semilinear elliptic equation with homogeneous Dirichlet boundary condition

\[-\text{div} A(x) D u = F(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad u \geq 0 \text{ in } \Omega,\]

where the nonlinearity \( F(x, u) \) is singular at \( u = 0 \), and more precisely where \( F \) is a Carathéodory function \( F : \Omega \times [0, +\infty] \to [0, +\infty] \) which satisfies

\[0 \leq F(x, s) \leq \frac{h(x)}{\Gamma(s)} \text{ a.e. } x \in \Omega, \forall s > 0,\]

with \( h \geq 0, h \in L^r(\Omega) \subset H^{-1}(\Omega) \) and \( \Gamma : [0, +\infty] \to [0, +\infty] \) a \( C^1 \), Lipschitz-continuous, nondecreasing function such that \( \Gamma(0) = 0 \) and \( \Gamma(s) > 0 \) for every \( s > 0 \). A model for such a function \( F(x, s) \) is for example given by

\[F(x, s) = \frac{f(x)}{\exp\left(-\frac{1}{s}\right)} \left( 2 + \sin\left(\frac{1}{s}\right) \right) + \frac{g(x)}{s^\gamma} + l(x) \text{ a.e. } x \in \Omega, \forall s > 0,\]

where the functions \( f, g \) and \( l \) are nonnegative and belong to \( L^r(\Omega) \).

The main difficulty is to give a convenient definition of the solution of this problem, in particular when \( \Gamma(s) \ll s \) for \( s \) close to 0.

We give such a definition and we prove the existence and stability of this solution, as well as its uniqueness when \( F(x, s) \) is non increasing in \( s \).

This work has been inspired by the papers [2] of Lucio Boccardo and Luigi Orsina and [1] of Lucio Boccardo and Juan Casado-Diaz.

References:


