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Synchronization in the random-field Kuramoto model on complex networks

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INTRODUCTION

The Kuramoto model [1] describes an emergence of collective synchrony of interacting oscillators. Depending on the coupling strength between them, the oscillators may be in a disordered state or in an ordered state. Most works on the Kuramoto model have described continuous phase transitions. Our previous findings [2] on the critical behavior of the order parameter, relaxation rate, and susceptibility of the Kuramoto model on uncorrelated random complex networks demonstrated that this model has the same critical exponents as the Ising model, and therefore it should belong to the same class of universality. This was shown for the Kuramoto model in the presence of a uniform external field. More generally, one should also analyze the impact of random fields (random pinning) on the critical behavior as in the random field Ising model on complex networks. The consideration of random fields may also be crucial to understand how disorder affects the emergence of synchrony in real systems such as cortical oscillations or the circadian clock in the brain. We present an analytical and numerical treatment of the Kuramoto model with heterogeneous random fields in random complex networks. We demonstrate that the network topology and the random field heterogeneity have a strong impact both on the critical coupling and on the kind of synchronization phase transition, which can be of both first- and second-order or infinite-order.

RANDOM FIELD KURAMOTO MODEL ON COMPLEX NETWORKS

The random field Kuramoto model describes the evolution of N phase oscillators according to the following equations:

$$\frac{d\theta_i}{dt} = \omega_i + K\sum_{j=1}^{N} a_{ji}\sin(\theta_j - \theta_i) + h_i \sin(\phi_i - \theta_i) \quad (1)$$

where θ_i is the phase of oscillator i, ω_i is its natural frequency, and K is the coupling constant. The oscillators' natural frequencies are heterogeneous and follow a probability density function $g(\omega)$. To analyze this equation, one can use the so-called "annealed network" approximation, which replaces a random complex network by a weighted complete graph in which edge weights are the probabilities of connections between nodes in the original graph, $\langle a_{ij}\rangle = q_i q_j / N\langle q\rangle$, where q_i is the degree of node i and $\langle q\rangle$ is the mean degree. Then, the order parameter is redefined as the fraction of synchornized oscillators

$$z = re^{i\psi} \equiv \frac{1}{N\langle q\rangle}\sum_{j=1}^{N} q_j e^{i\theta_j} \quad (2)$$

The phase of the random local fields are uniformly distributed in (0, 2π]. Regarding the field magnitude, first we study the case of that all local fields have the same magnitude, i.e. $h_i = h$ and second the entries are Gaussian-distributed.

We use the Ott-Antonsen method [3] to find a set of differential equations for the time evolution of the order parameter z. We look for the stationary state solution at which the phase coherence r is constant. With suitable reference frame rotation ($\omega \mapsto \omega+\Omega$) and $\psi = 0$, that is, $z = r$, we obtain a nonlinear equation determining r as a function of K, degree, and random field distributions,

$$r = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} G(h)dh \int_1^{\infty} \frac{qP(q)}{\langle q\rangle} \frac{Kqr + h\cos\phi}{\sqrt{(Kqr + h\cos\phi)^2 + h^2\sin^2\phi + 1} + 1} dq \quad (3)$$

SUMMARY

We studied the impact of random pinning fields on the synchronization of phase oscillators in the Kuramoto model on a complete graph and uncorrelated complex networks with different degree distributions (scale-free networks).

1. For homogeneous random fields, 1) in the fully connected network and scale-free networks with the degree exponent γ > 5, there is a critical random-field magnitude above which the second-order phase transition gives place to a first-order phase transition. 2) In scale-free networks with 2 < γ ≤ 5, the phase transition remains of second-order at any random-field magnitude. In the case γ = 3, the synchronization transition is of infinite order. (See Table 1)

2. With heterogeneous random fields (Gaussian random fields), the critical coupling depends strongly on the field variance. The continuous phase transition into the synchronized state is characterized by the same critical exponents as the synchronization transition in the absence of the fields. (See Table 1)

CRITICAL BEHAVIOR

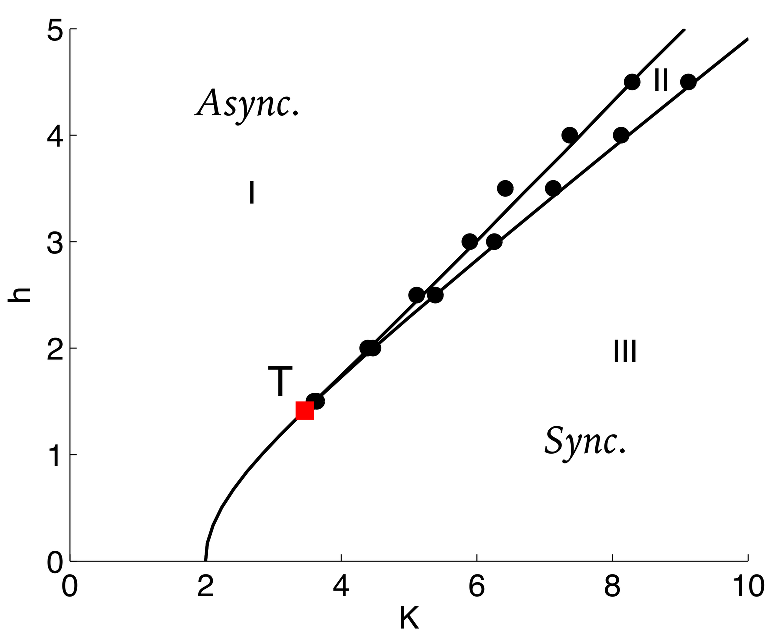
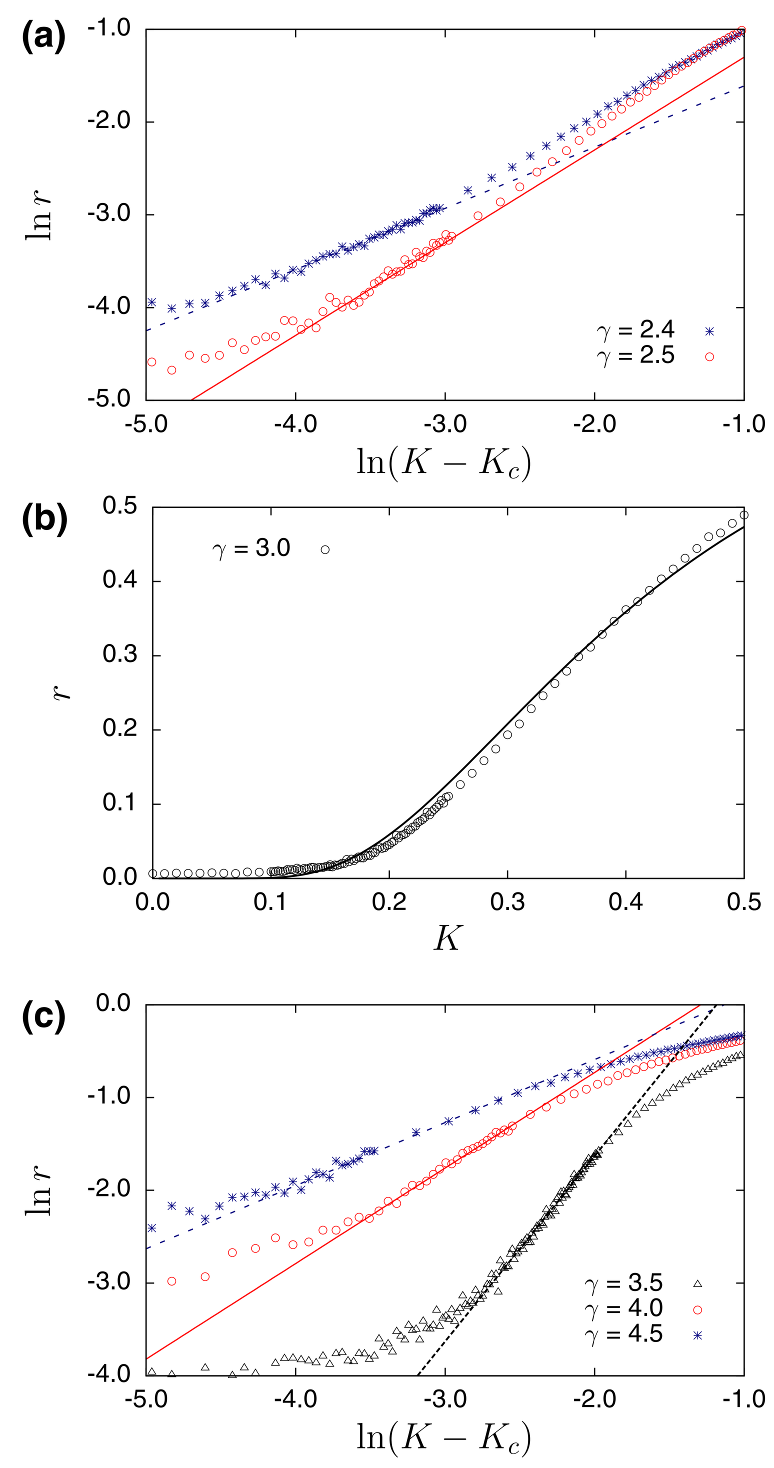


FIGURE1 : *K-h* phase diagram of the Kuramoto model on a complete graph in the presence of homogeneous random fields of magnitude *h*. The solid line corresponds to the numerical solution of Eq. (3) and the points to our simulations. There are three regions: (I) asynchronous state, (II) region of hysteresis, and (III) partially synchronous state. $T = (K_c, h_c)$ is a tricritical point below which there is a second-order phase transition, whereas above it there is a first-order phase transition.

Network	Field: Homogeneous	Field: Gaussian
Complete	$r \propto \sqrt{K-K_c}$ (*)	
Graph	$K_c = 2\sqrt{h^2+1}$	$K_c = f(\sigma)$
Scale-free	$r \propto \sqrt{K-K_c}$ (*)	
$\gamma > 5$	$K_c = 2\sqrt{h^2+1}\frac{\langle q\rangle}{\langle q^2\rangle}$	$K_c = f(\sigma)\frac{\langle q\rangle}{\langle q^2\rangle}$
$3 < \gamma \leqslant 5$	$r \propto (K-K_c)^{1/(\gamma-3)}$	
	$K_c = 2\sqrt{h^2+1}\frac{\langle q\rangle}{\langle q^2\rangle}$	$K_c = f(\sigma)\frac{\langle q\rangle}{\langle q^2\rangle}$
$\gamma = 3$	$r \propto \frac{1}{K}e^{-1/K}$	
	$K_c = 0$	
$2 < \gamma < 3$	$r \propto (K-K_c)^{(2-\gamma)/(\gamma-3)}$	
	$K_c = 0$	

TABLE 1 : Critical behavior of the order parameter of the Kuramoto model on a complete graph and complex networks with degree distribution $P(q) \propto q^{-\gamma}$ in the presence of homogeneous and Gaussian random fields. In the case of a complete graph and scale-free network with γ > 5 (*), the presented critical behavior occurs in the case of Gaussian fields and homogenous fields at $h < \sqrt{2}$. At $h > \sqrt{2}$, the transition is discontinuous.



γ	β (A)	β (S)	K_c (A)	K_c (S)
2.4	$\frac{(2-\gamma)}{(\gamma-3)} \approx 0.667$	0.66 ± 0.01	0	0.052
2.5	$\frac{(2-\gamma)}{(\gamma-3)} = 1$	1.00 ± 0.03	0	0.048
3.5	$\frac{1}{(\gamma-3)} = 2$	2.00 ± 0.02	$2\sqrt{1+h^2}\frac{\langle q\rangle}{\langle q^2\rangle} \approx 0.266$	0.211
4.0	$\frac{1}{(\gamma-3)} = 1$	1.03 ± 0.02	$2\sqrt{1+h^2}\frac{\langle q\rangle}{\langle q^2\rangle} \approx 0.319$	0.322
4.5	$\frac{1}{(\gamma-3)} \approx 0.667$	0.68 ± 0.02	$2\sqrt{1+h^2}\frac{\langle q\rangle}{\langle q^2\rangle} \approx 0.349$	0.369

TABLE 2 : Comparison of the critical coupling K_c and the order parameter exponent β between the simulations (S) in Fig. 2 and the analytical (A) results in Table 1 for random field magnitude $h = 2$.

FIGURE 2 : Critical behavior of the order parameter in scale-free networks in the presence of homogeneous random fields (h = 2). The symbols correspond to simulations of the model, whereas the lines in panels (a) and (c) are the linear fits. The line in panel (b) is $f(K) \propto 1/K \exp(-1/K)$.

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