A general notion of interiority

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MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

two issues...

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- □ to what extent a set of entities $\{\bullet\}$ is separated from $\bullet$?

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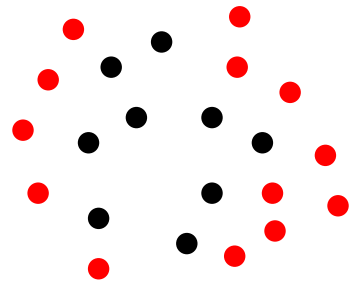

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- to what extent a set of entities {●} is separated from ● ?

two issues...

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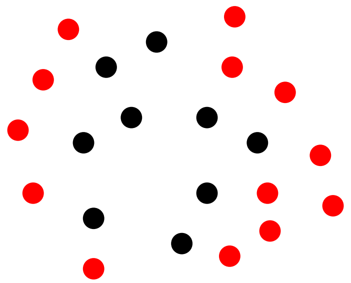

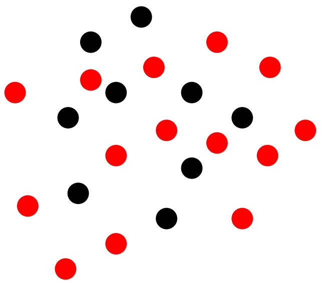

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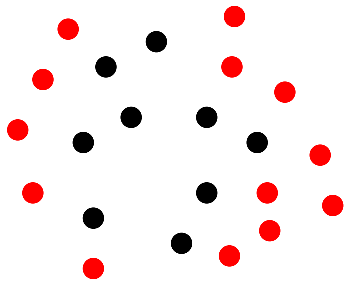

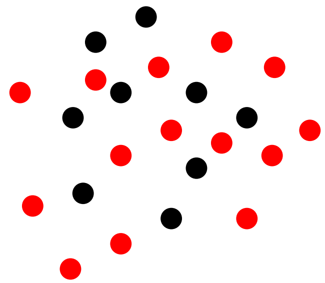

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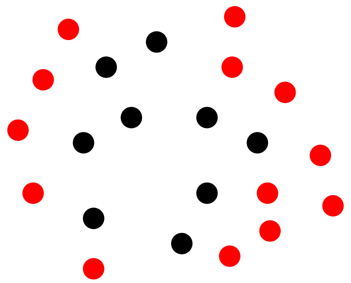

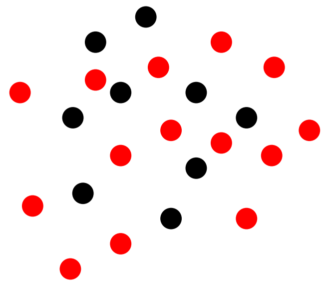

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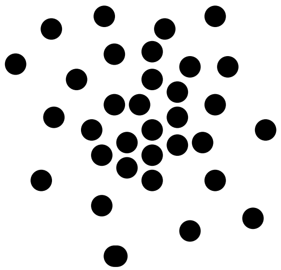

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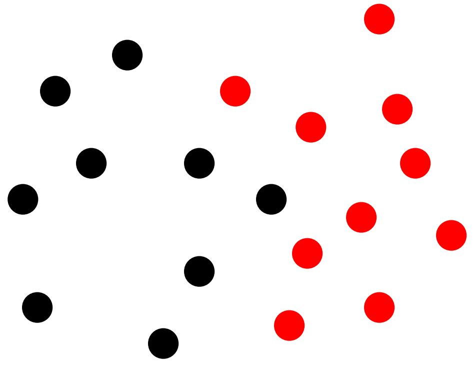

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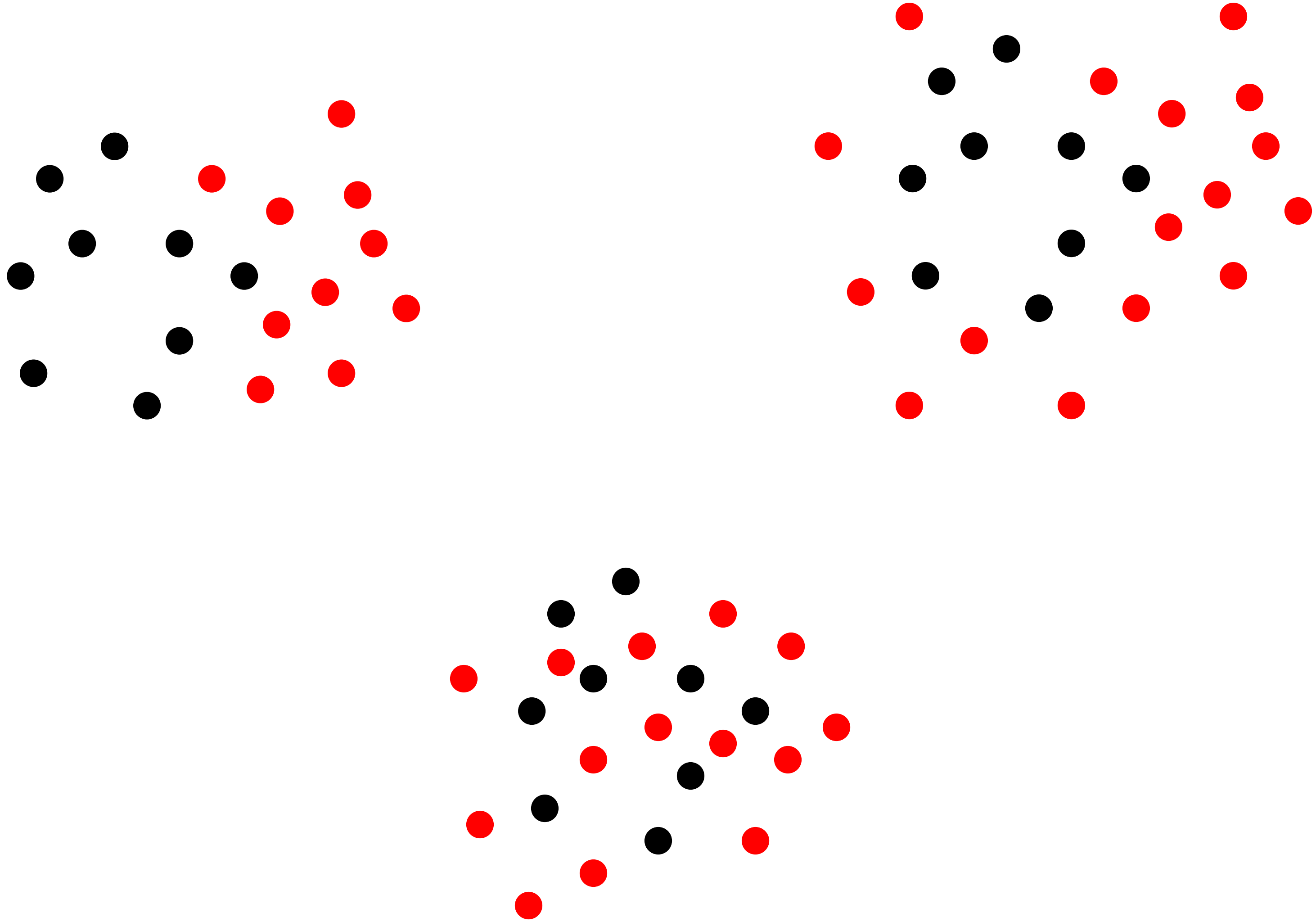

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- □ all, except Thornton's index, are expressions that explicitly involve numerical dissimilarities between entities
- □ notions such as "perfect separation of cluster" are not comprise.

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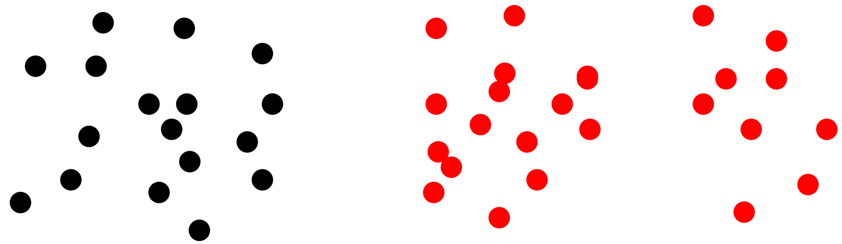

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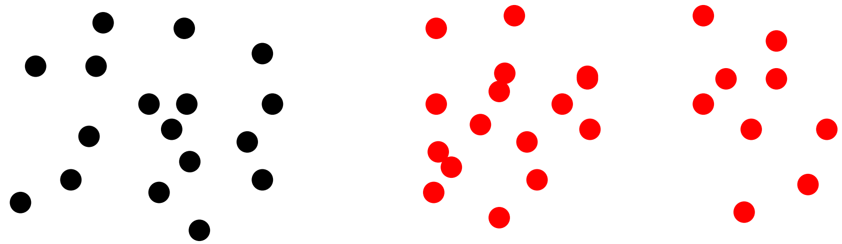

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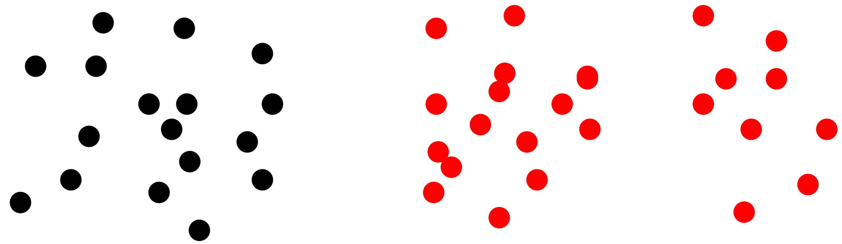

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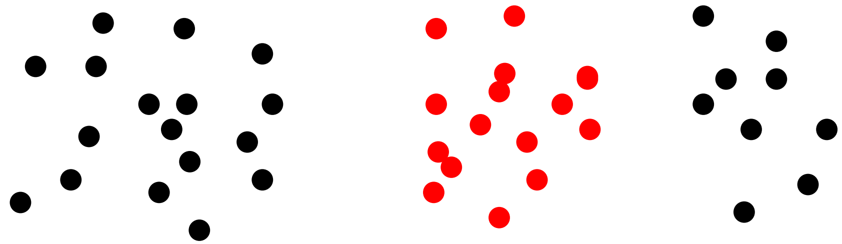

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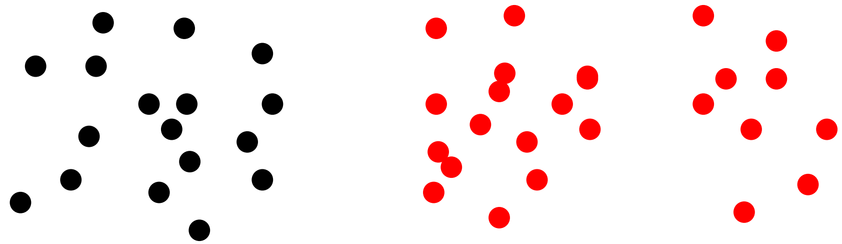

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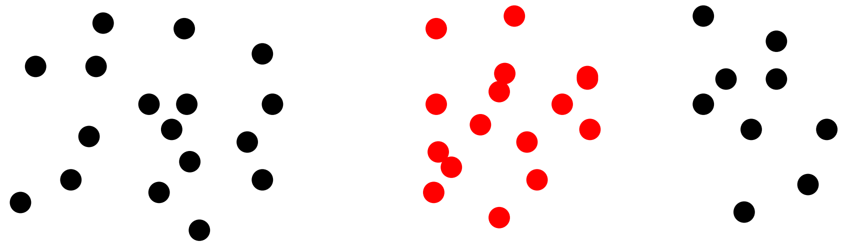

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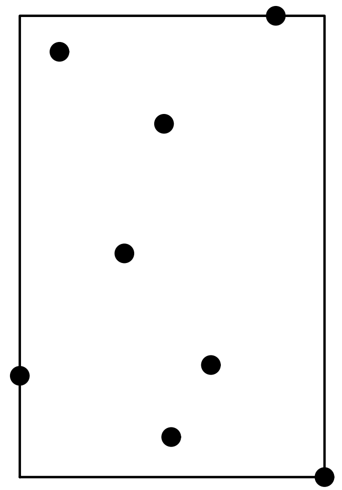

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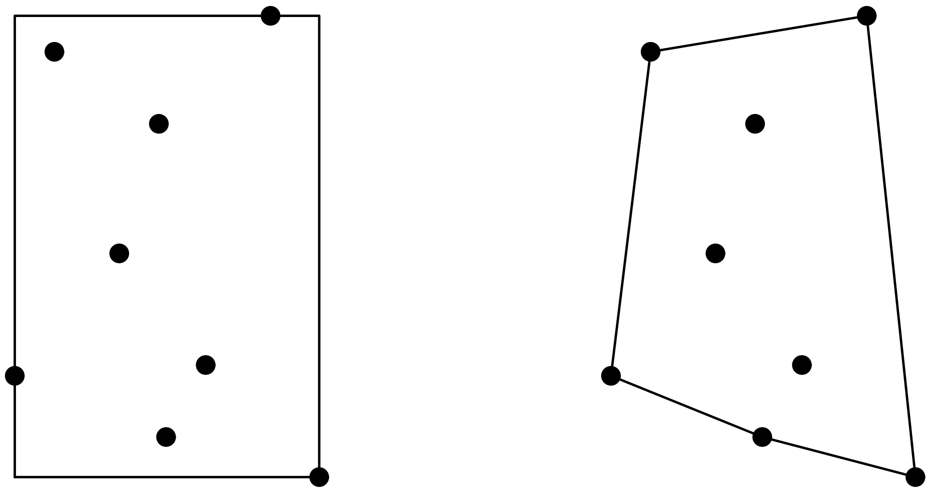

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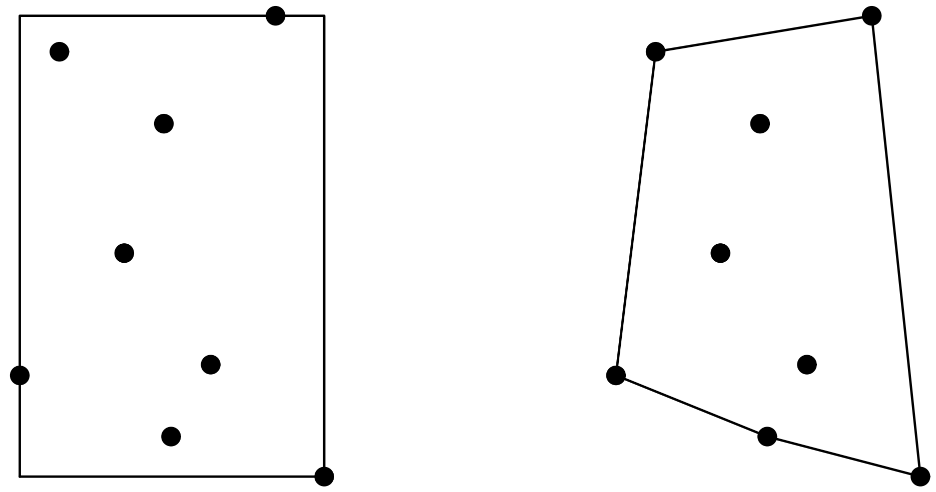

doesn't discriminate separability

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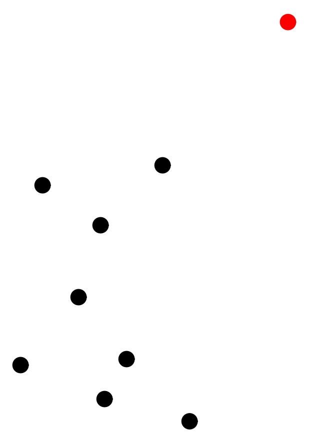

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▷ Introduction
Interiority
Separability index
Convexity criterion
Auto-interiority

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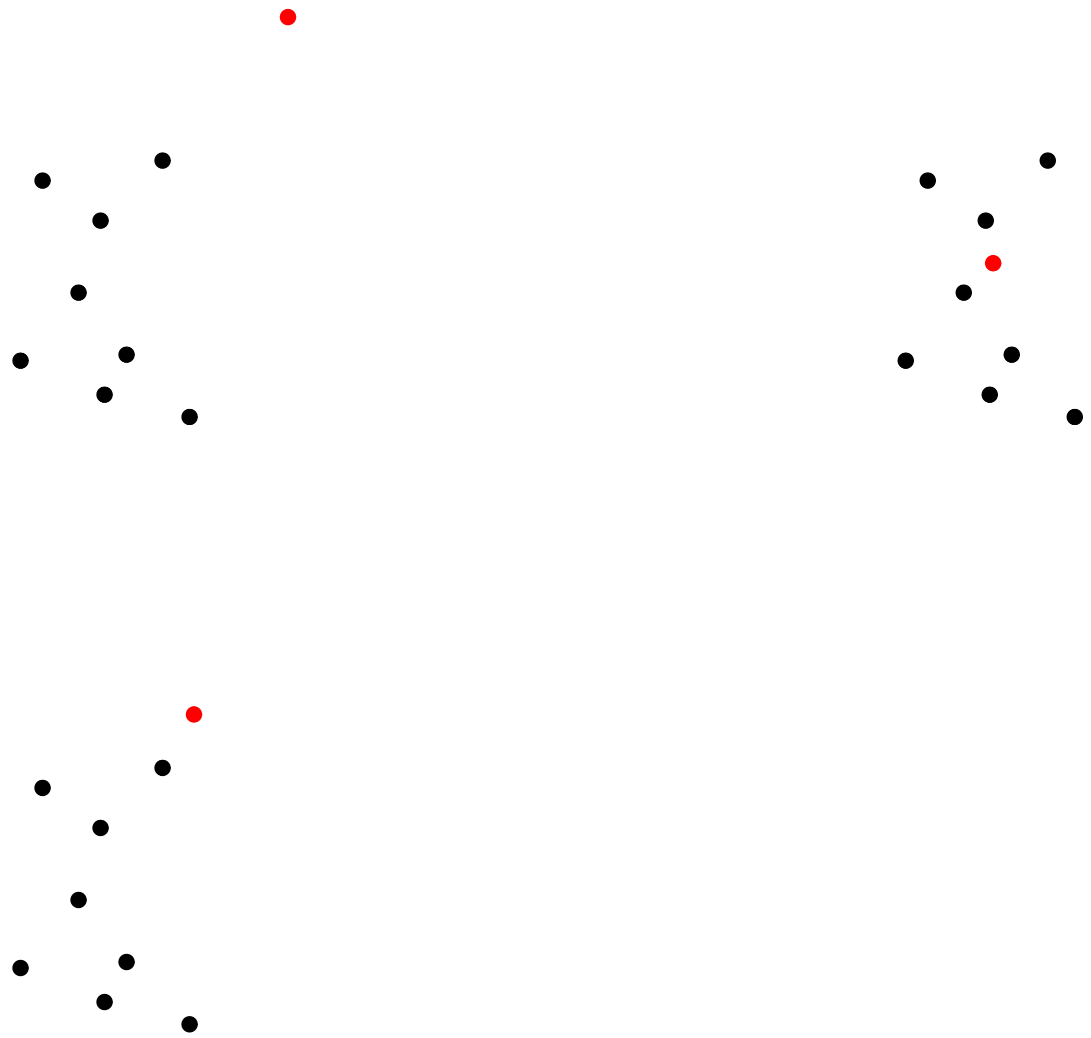

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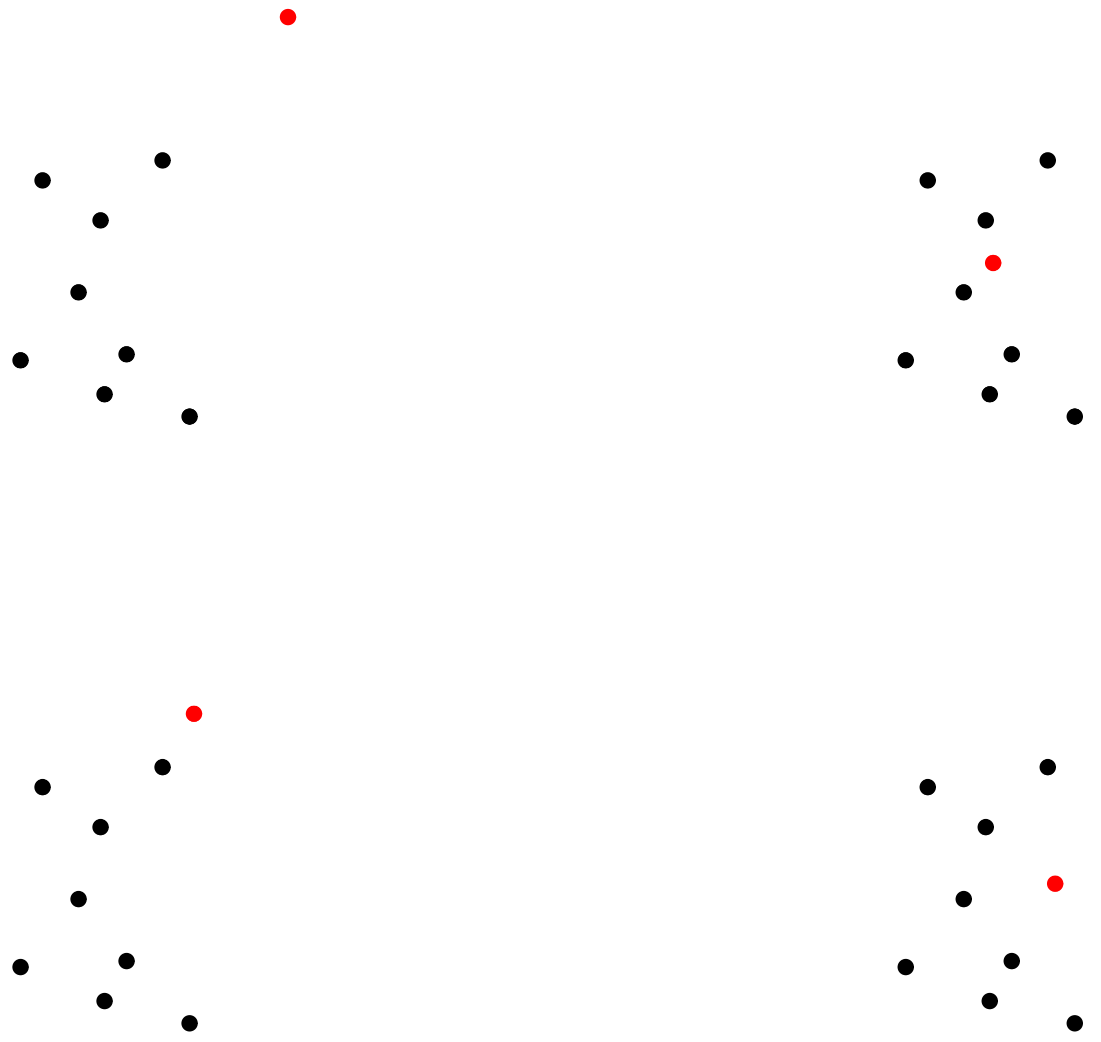

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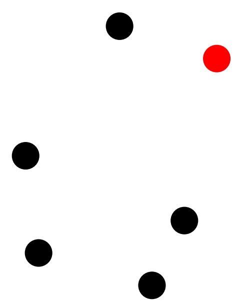

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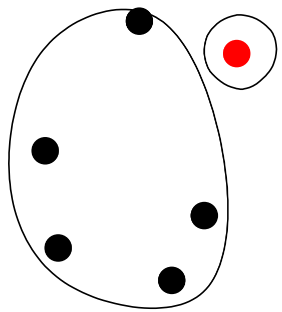

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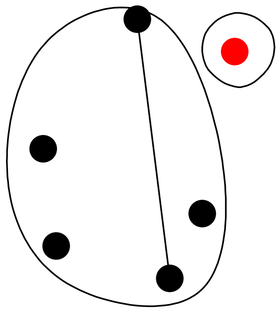

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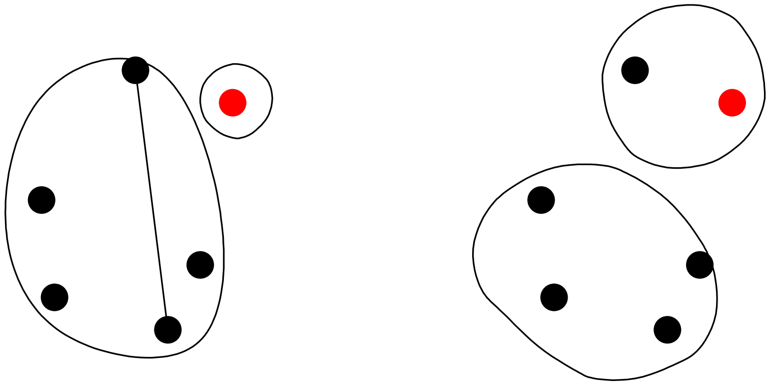

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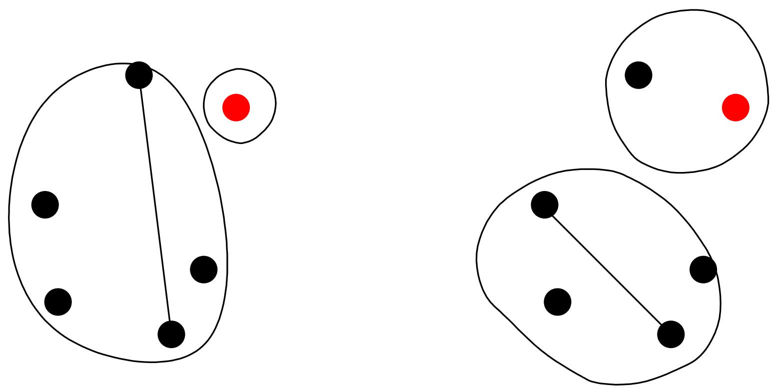

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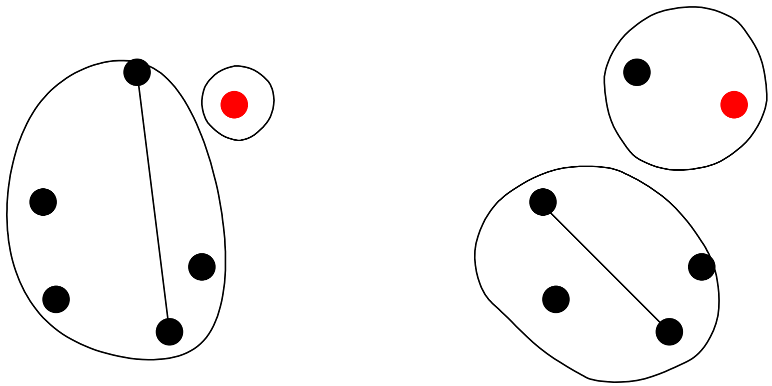

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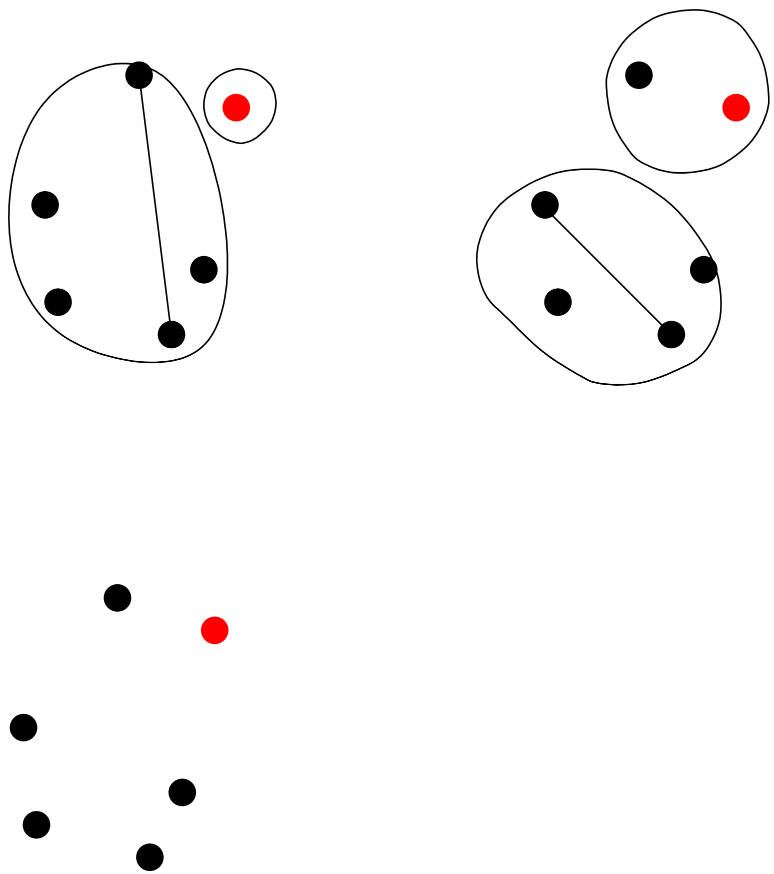

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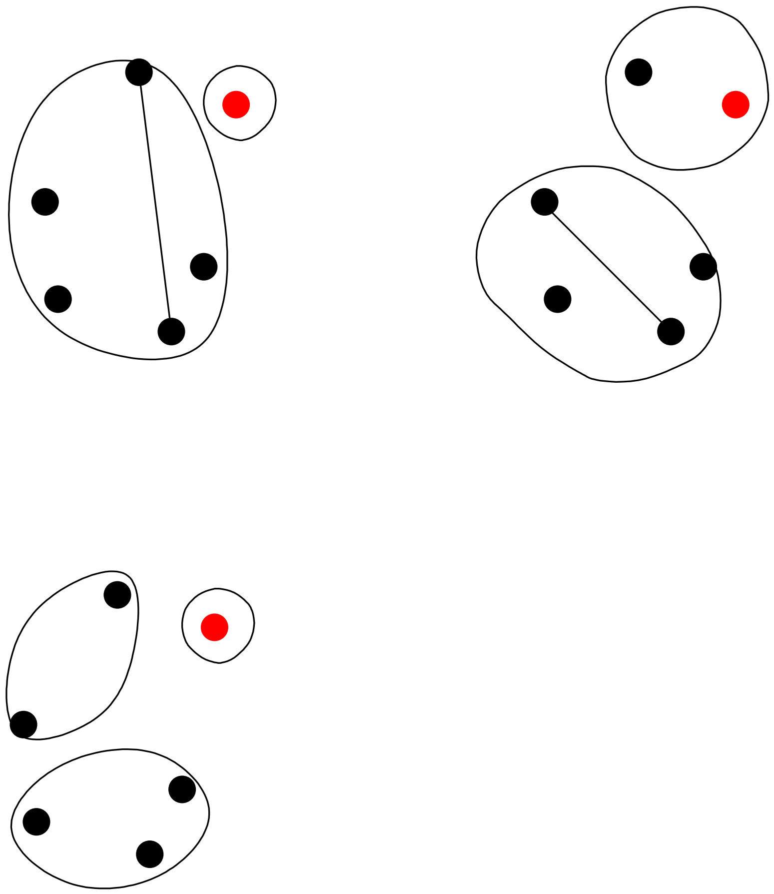

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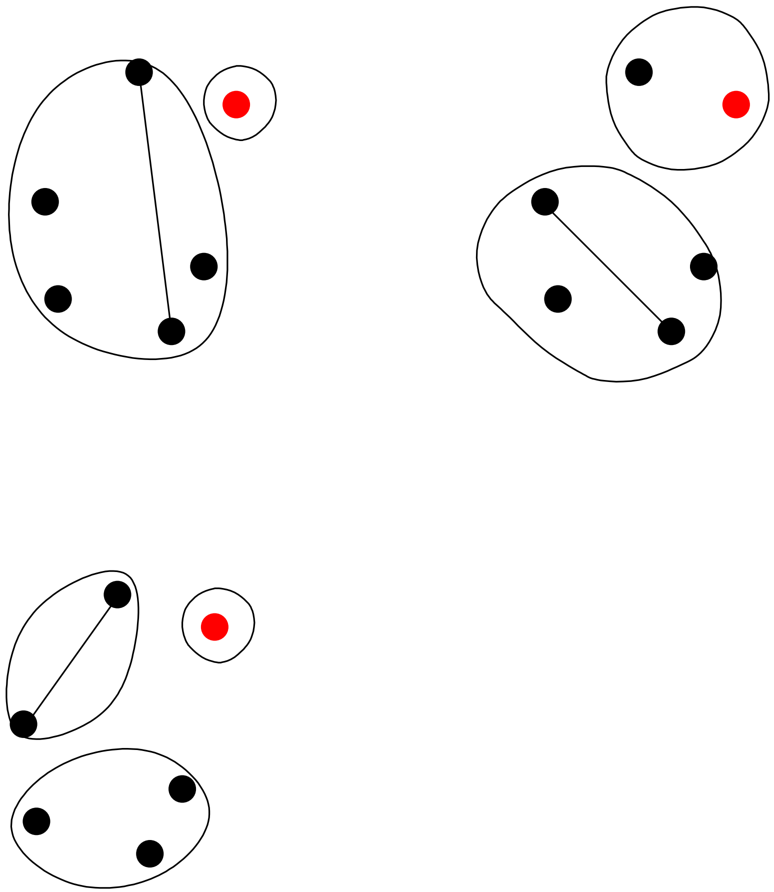

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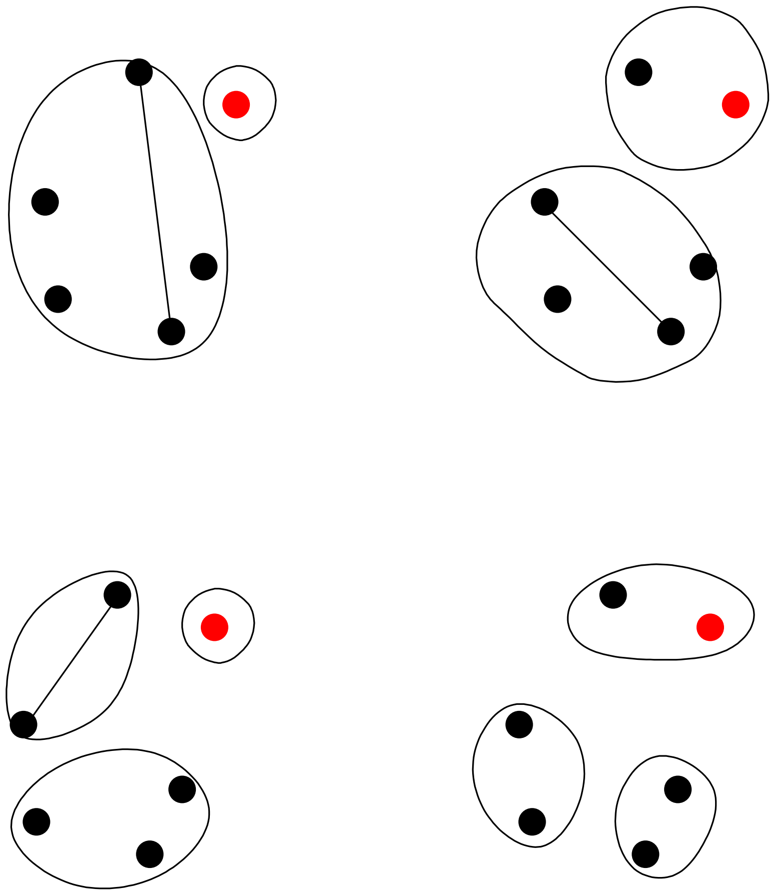

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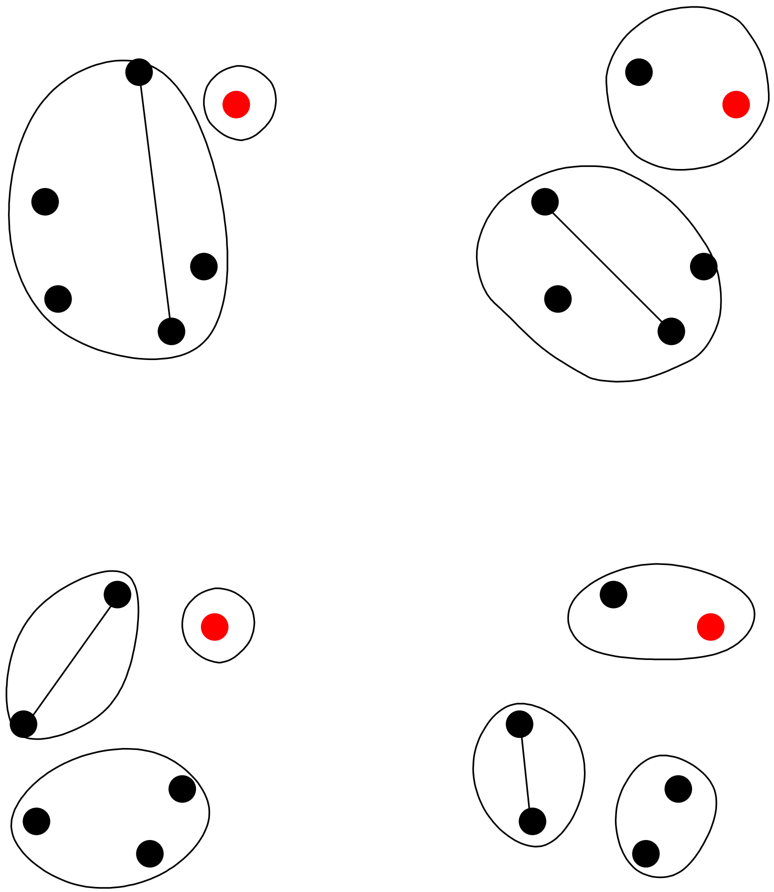

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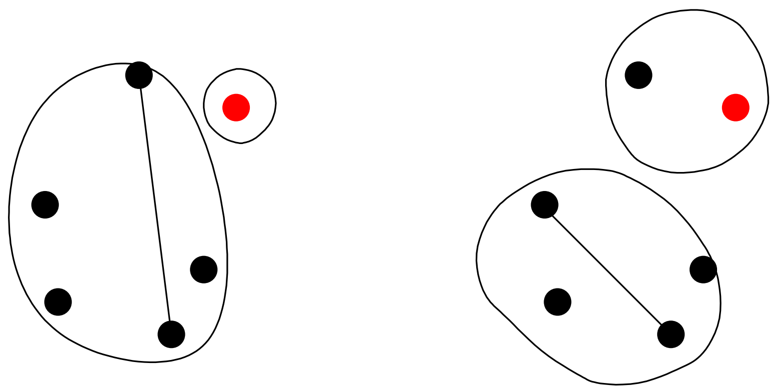
entity $\bar{x}$ is k-*interior* with respect to X if for every k-partition $\mathcal{P}_k$ of X, $\mathcal{P}_k \cup \{\{\bar{x}\}\}$ is not an adequate $(k+1)$-partition of $X \cup \{\bar{x}\}$



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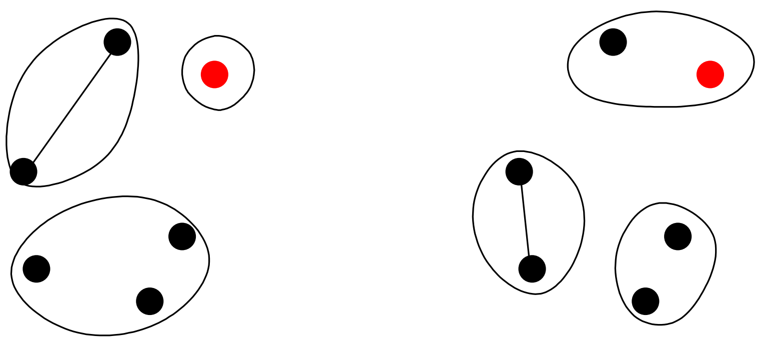

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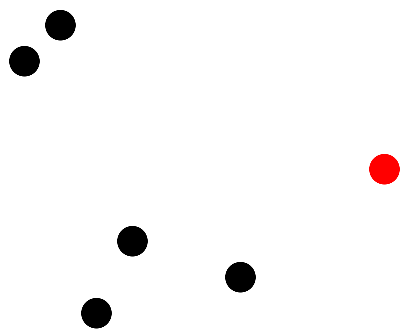

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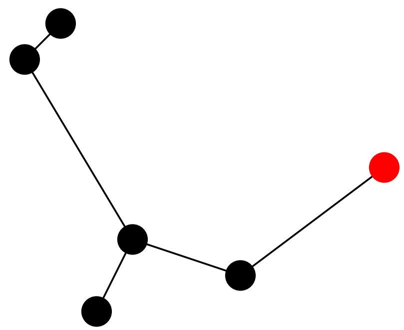

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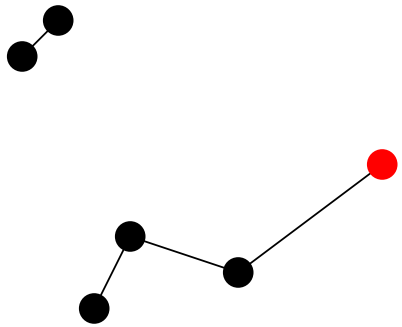

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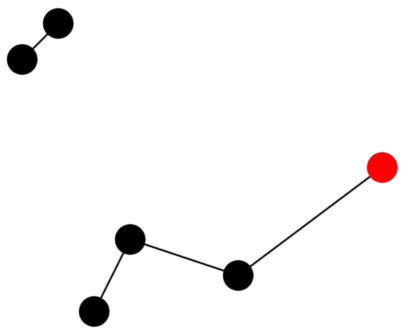

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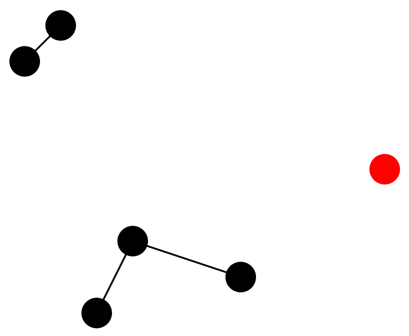

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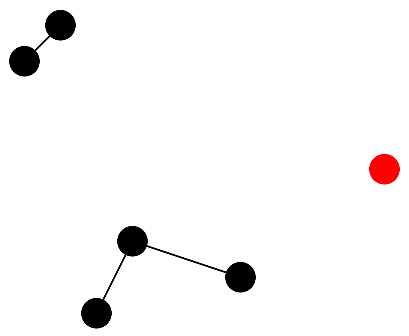

● is 1-interior   but not 2-interior

interiority degree

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$d_{\bar{x}} =$ the largest k such that $\bar{x}$ is k-interior

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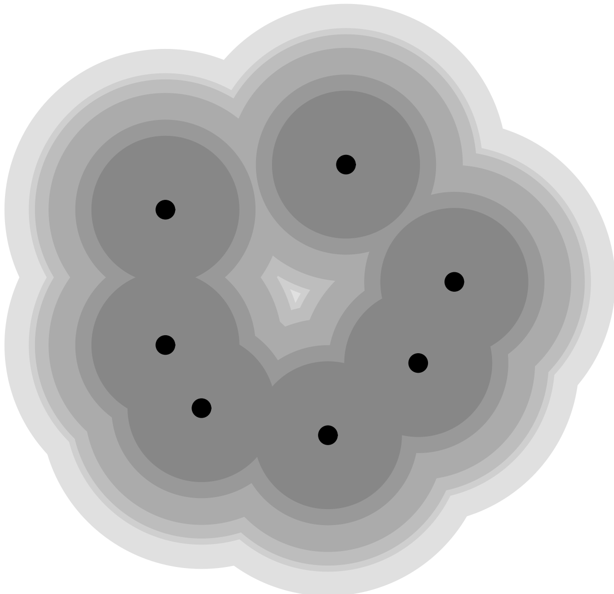
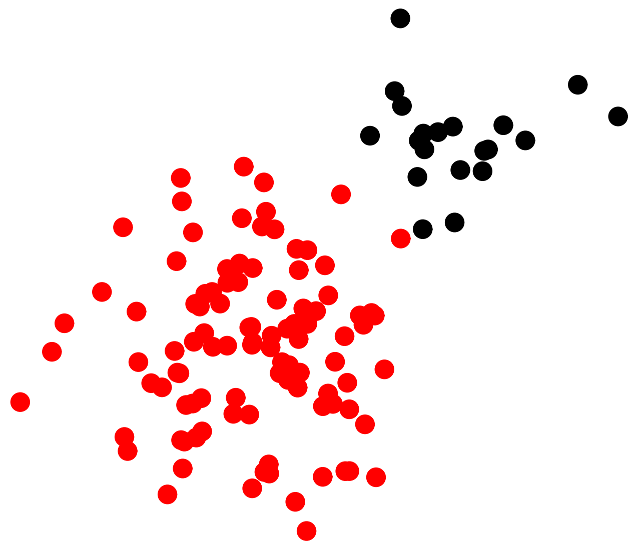


separability index



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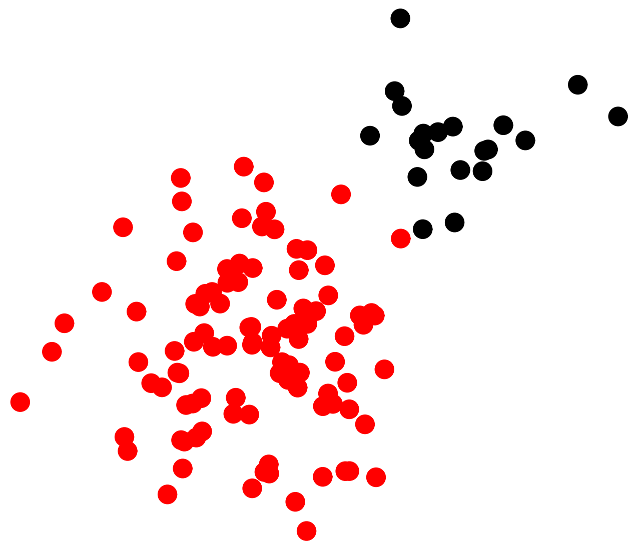

$$D_{\bar{X}} = \{d(\bar{x}) : \bar{x} \in \bar{X} = \{\bullet\}\}$$ *interiority* of $\bar{X}$

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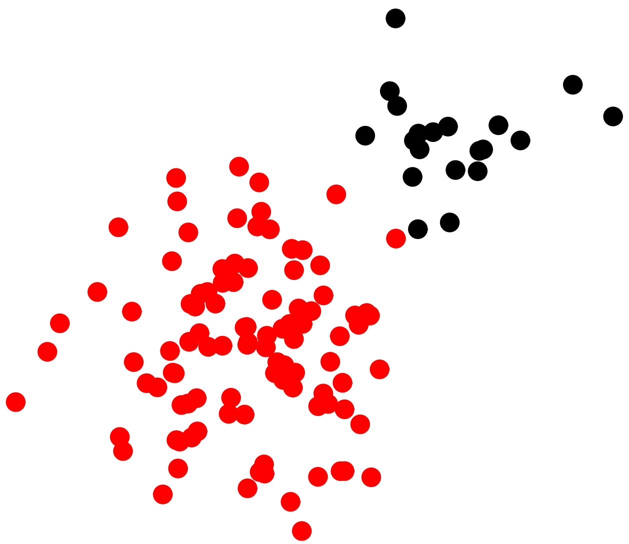

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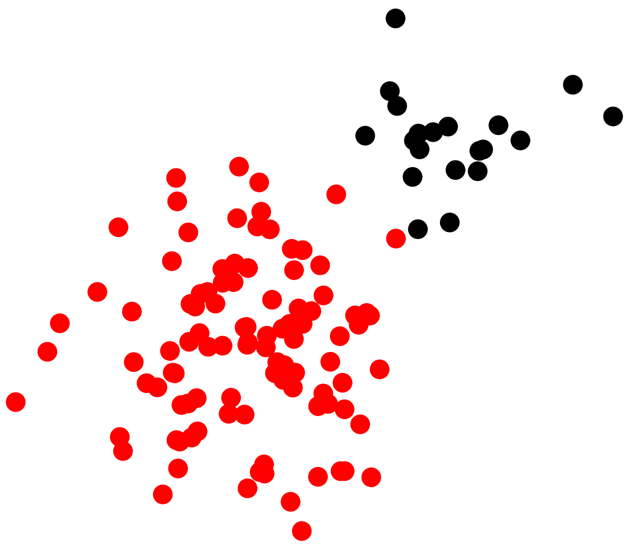

$D_{\bar{X}} = \{d(\bar{x}) : \bar{x} \in \bar{X} = \{\textcolor{red}{\bullet}\}\}$ *interiority* of $\bar{X}$

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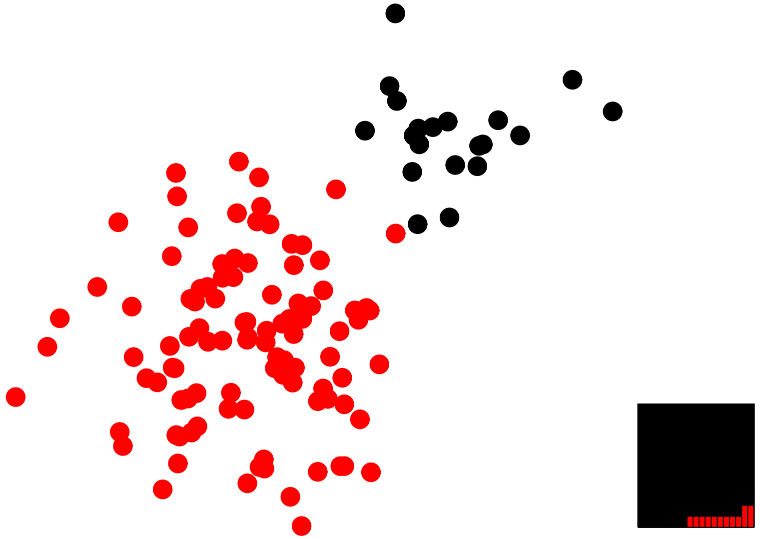
DSI of m points with interiority degree n = DSI of n points with interiority degree m

separability index



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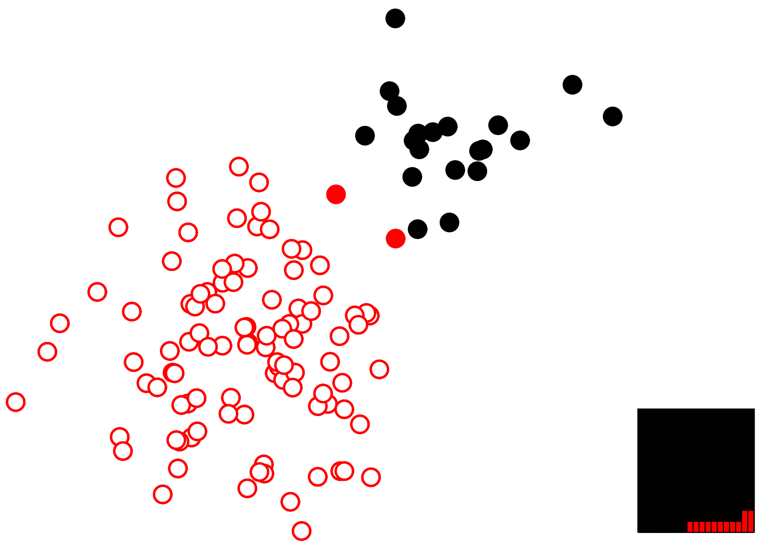

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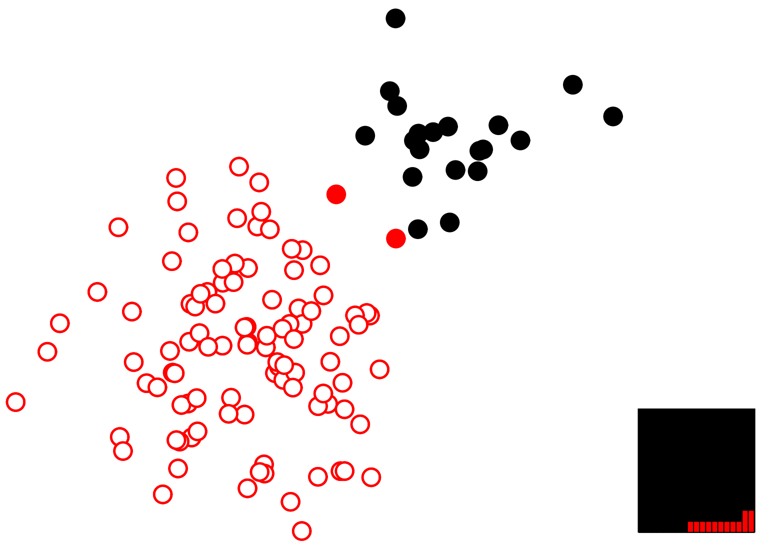

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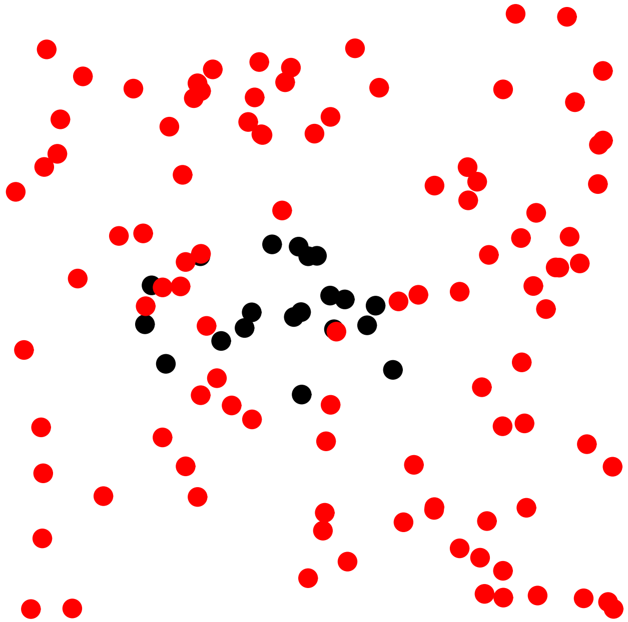

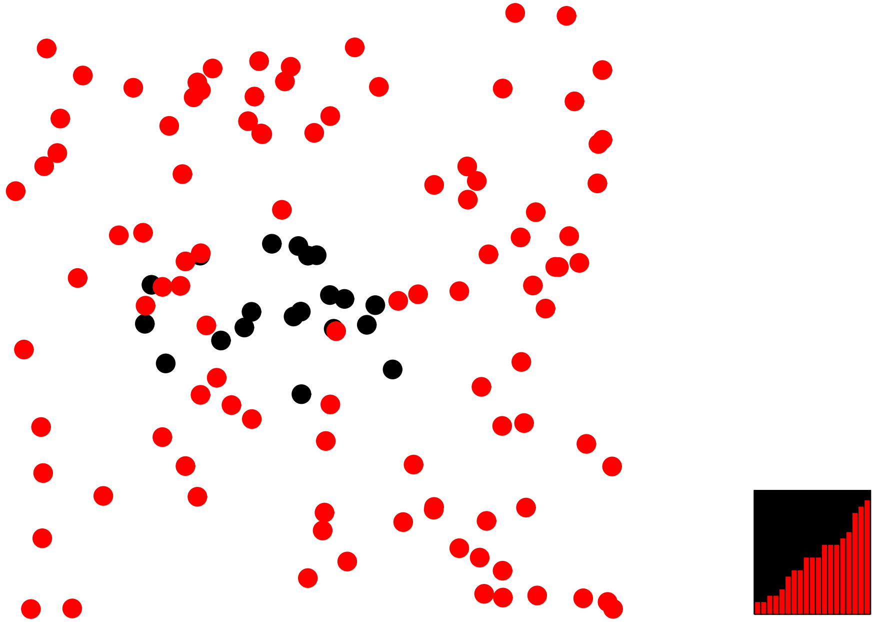

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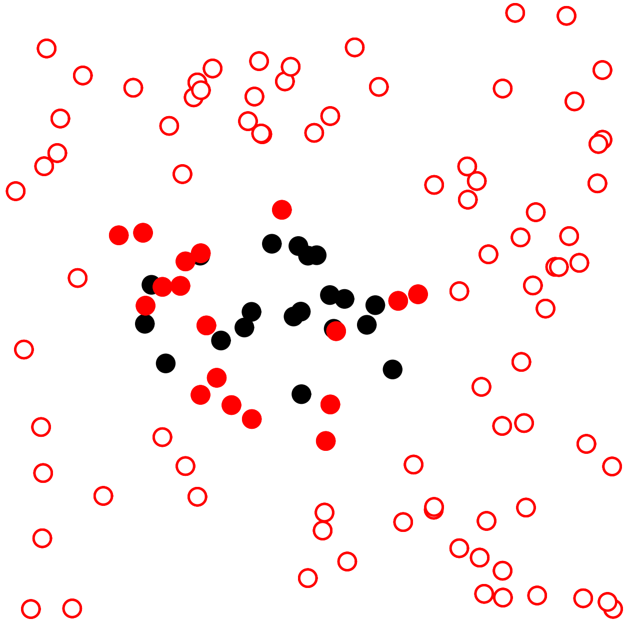

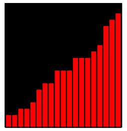

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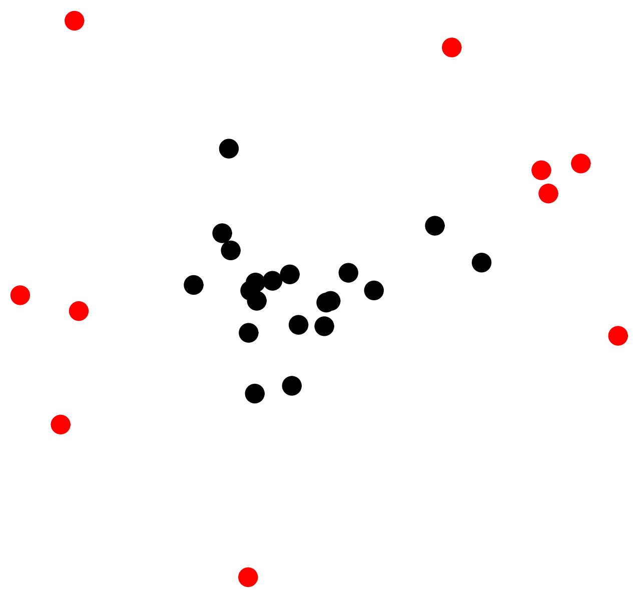

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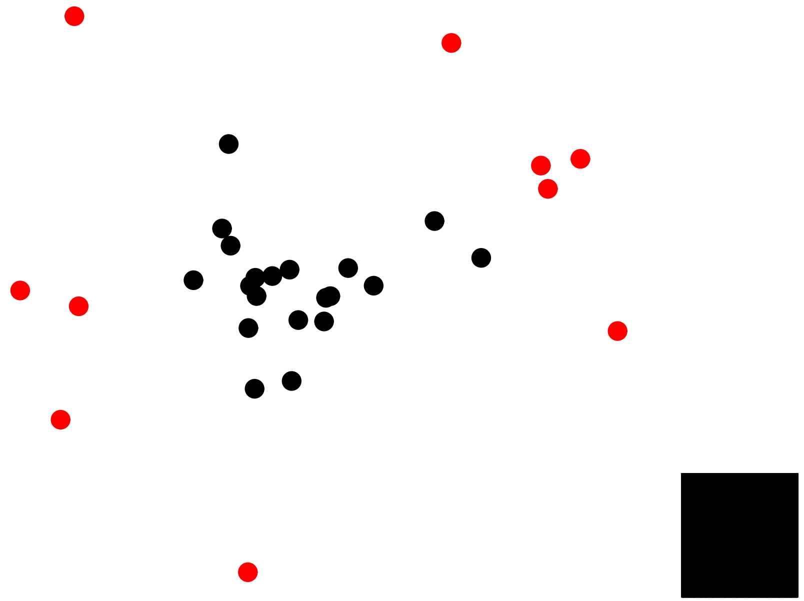

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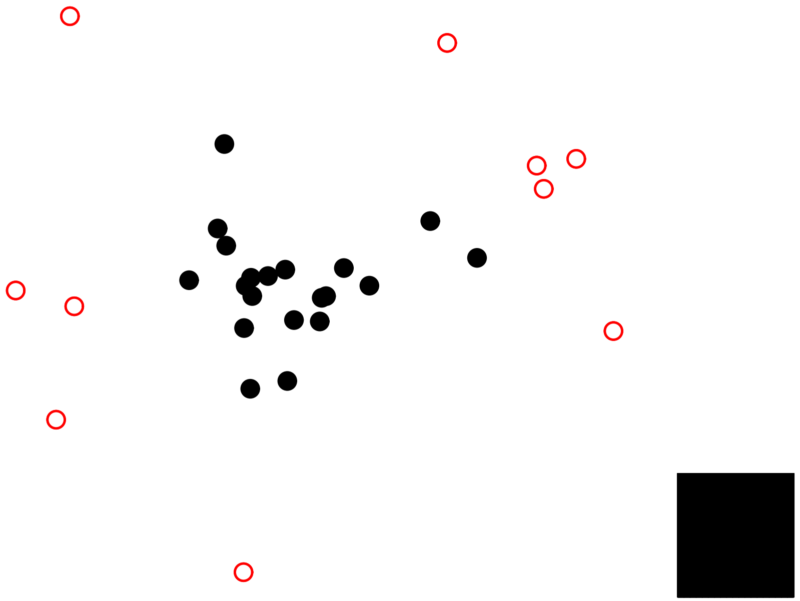

convexity criterion

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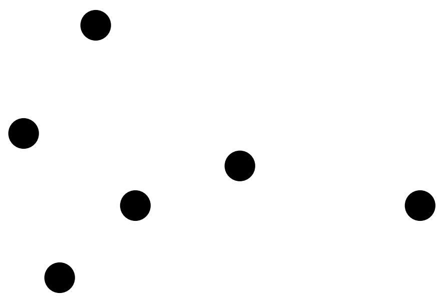

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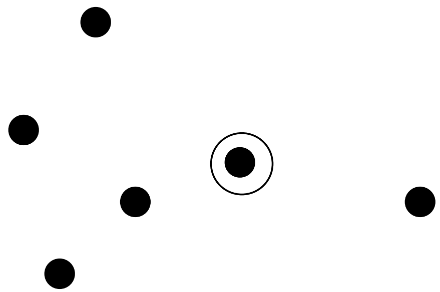

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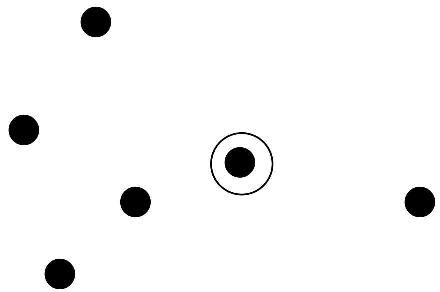

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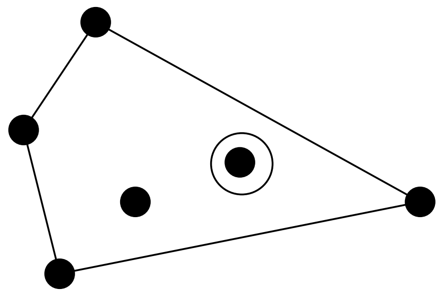

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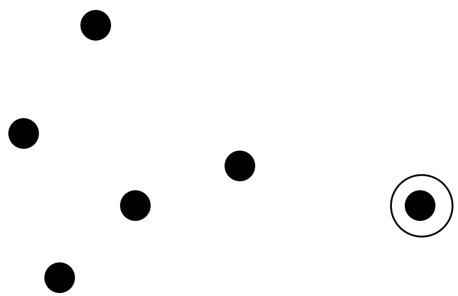

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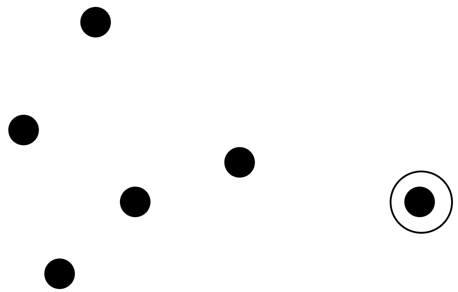

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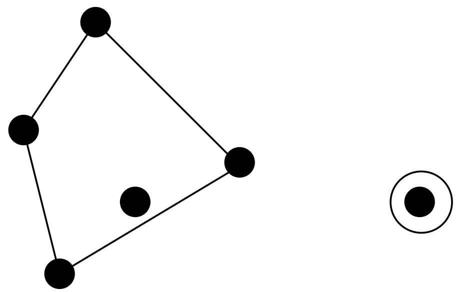

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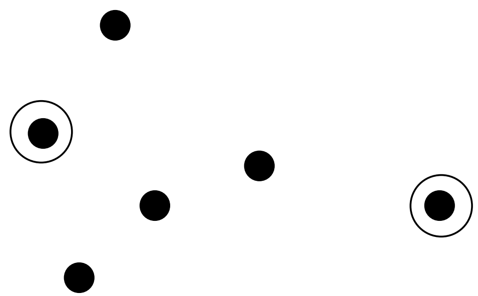

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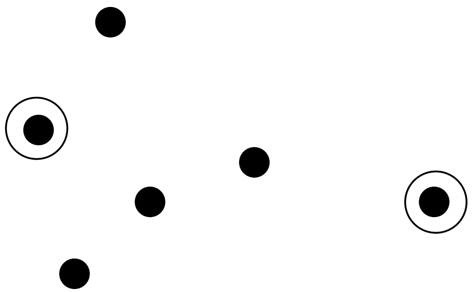

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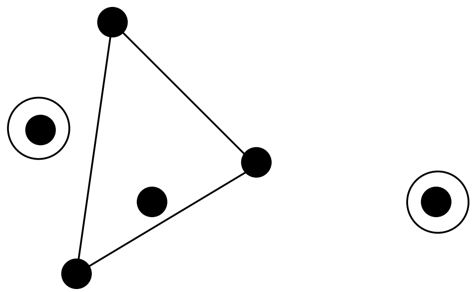

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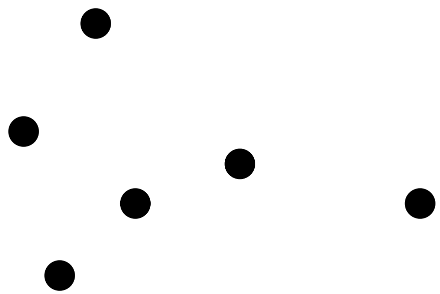

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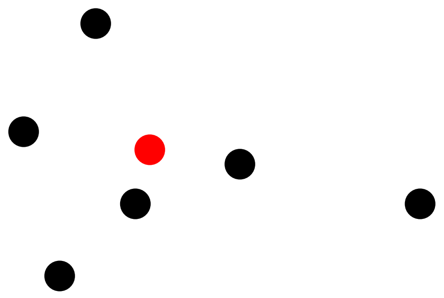

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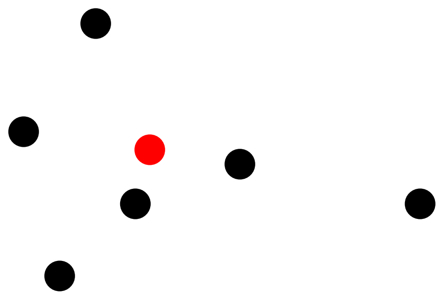

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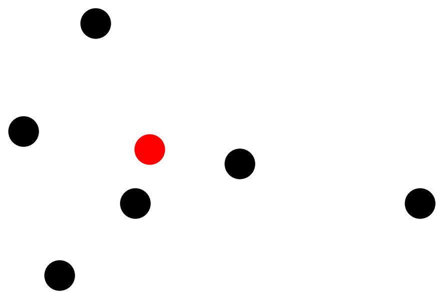

● is 1-interior, and 2-interior

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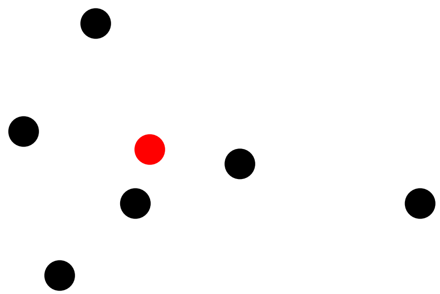

● is 1-interior, and 2-interior

$d(\bullet)$ is the minimum number of points to be removed from X such that ● is outside conv. hull of the remaining points

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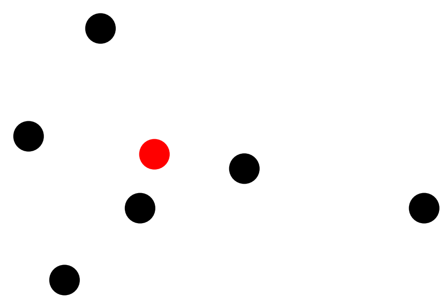

● is 1-interior, and 2-interior

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$d(\bullet) = 2$

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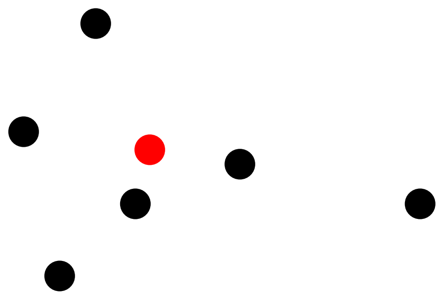

$\bullet$ is 1-interior, and 2-interior

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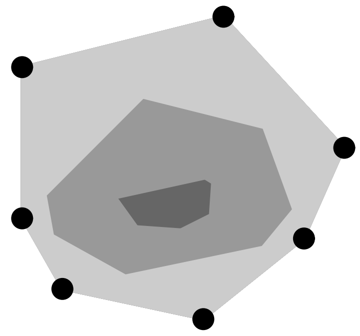
$d(\bullet) = 2$

$d(\ )$ - *(halfspace, location, Tukey) depth*

iso-interiority regions



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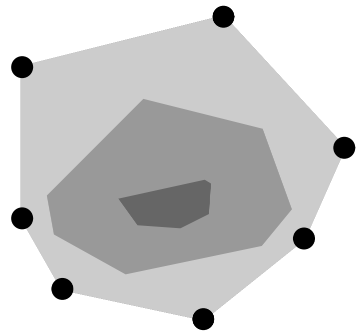

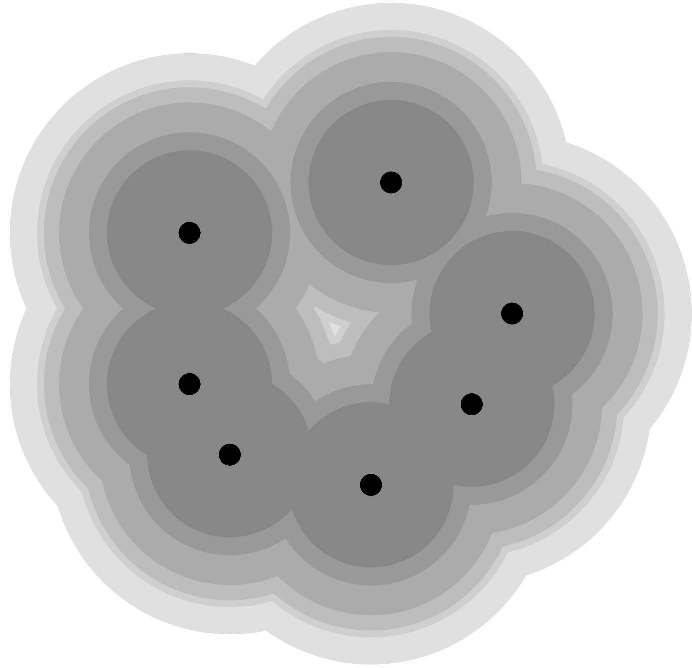

depth $\hookrightarrow$ auto-interiority

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$D_X = \{d(x) : x \in X\}$ *depth* of X

depth $\hookrightarrow$ auto-interiority

for $x \in X$, $d(x)$ is the minimum number of points to be removed from X such that x is outside the conv.hull of the remaining points

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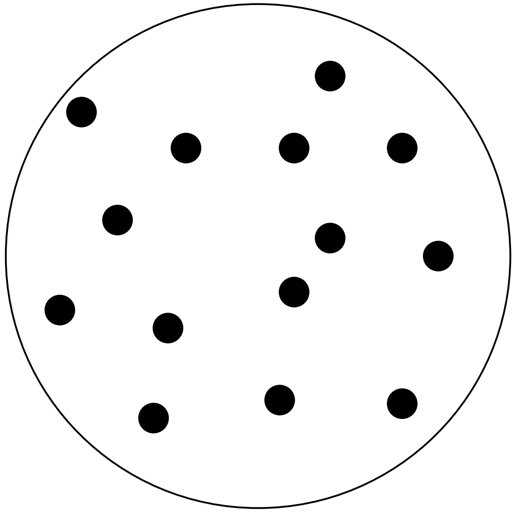
if X is uniformly distributed on an n-ball



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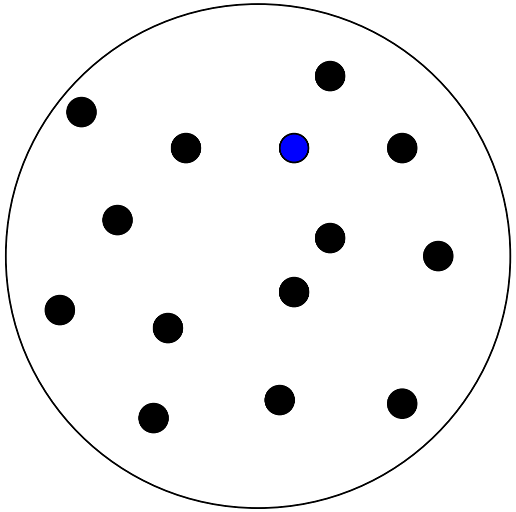
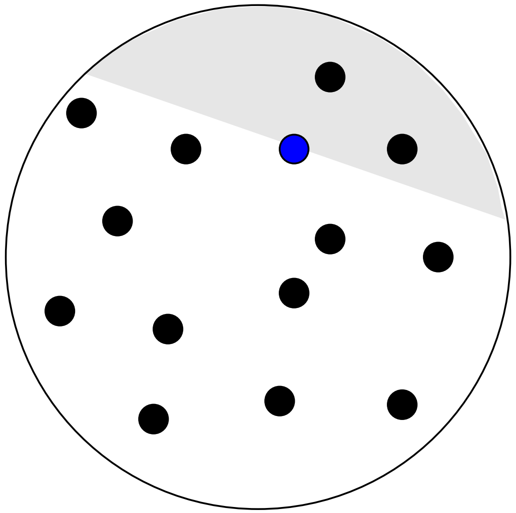


$$\frac{d(x)}{|X|} \rightsquigarrow$$



$$\frac{d(x)}{|X|} \rightsquigarrow \frac{vol(\;)}{vol(B_n)} = V_r(x)$$

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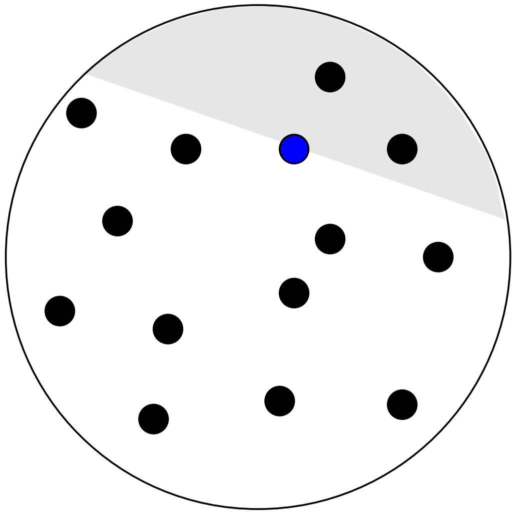

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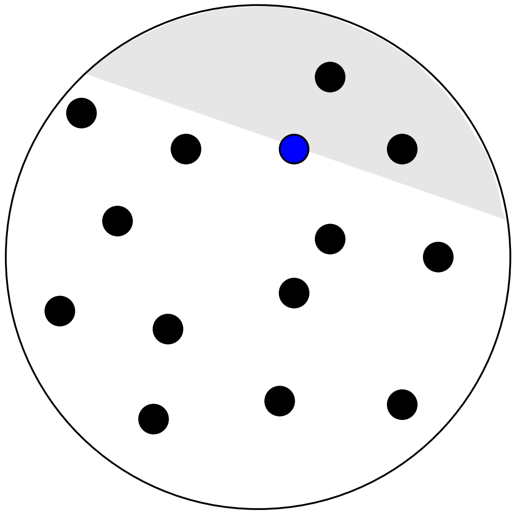

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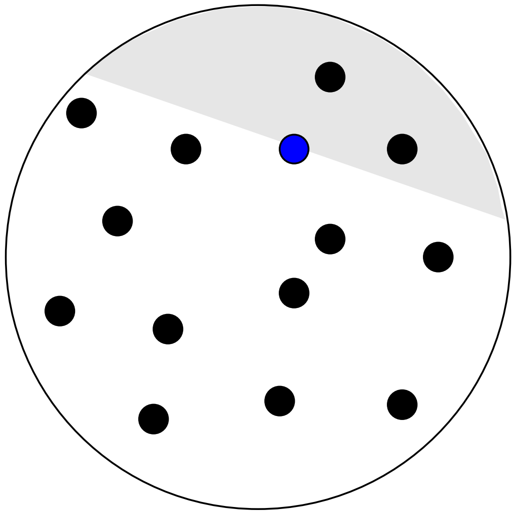

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Result holds if X is uniformly distributed on the closed region of $\mathbb{R}^n$ delimited by an ellipsoid

if X is uniformly distributed on an n-ball

$$E[V_r(x)] = \frac{1}{2^{n+1}}$$

if X is uniformly distributed on an n-ball

$$E[V_r(x)] = \frac{1}{2^{n+1}} \quad \Longrightarrow \quad \frac{1}{|X|}\sum_x V_r(x) \approx \frac{1}{2^{n+1}}$$

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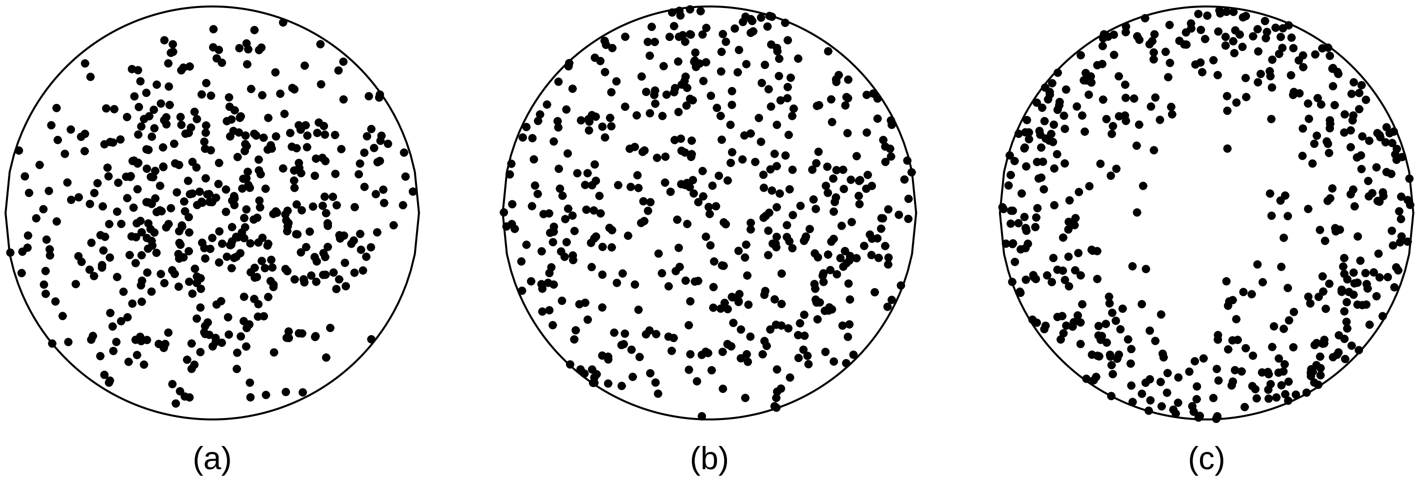
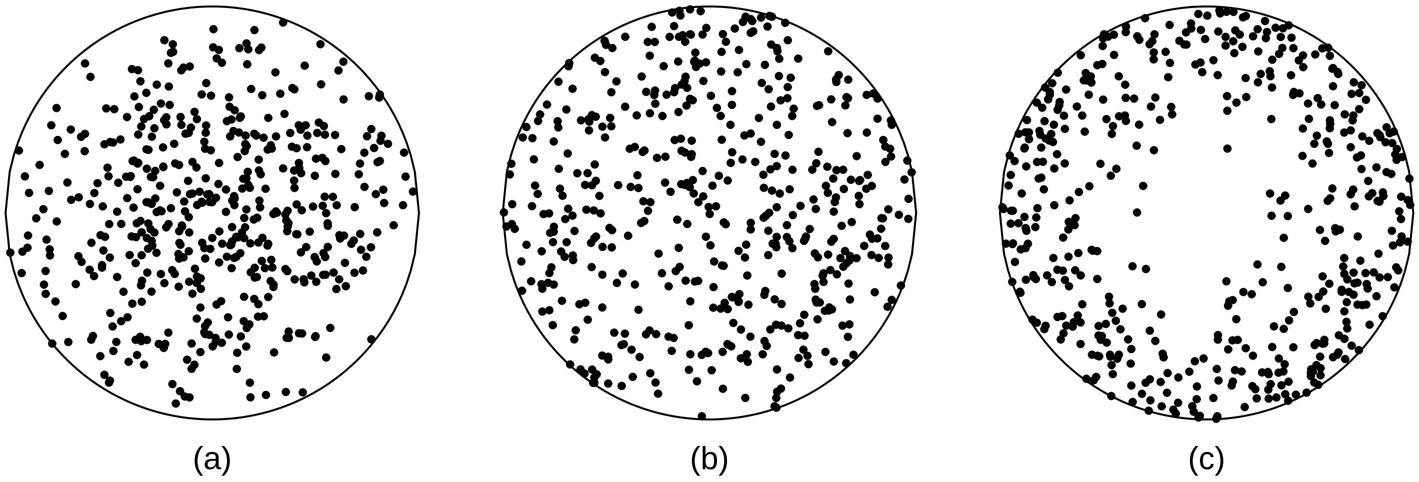


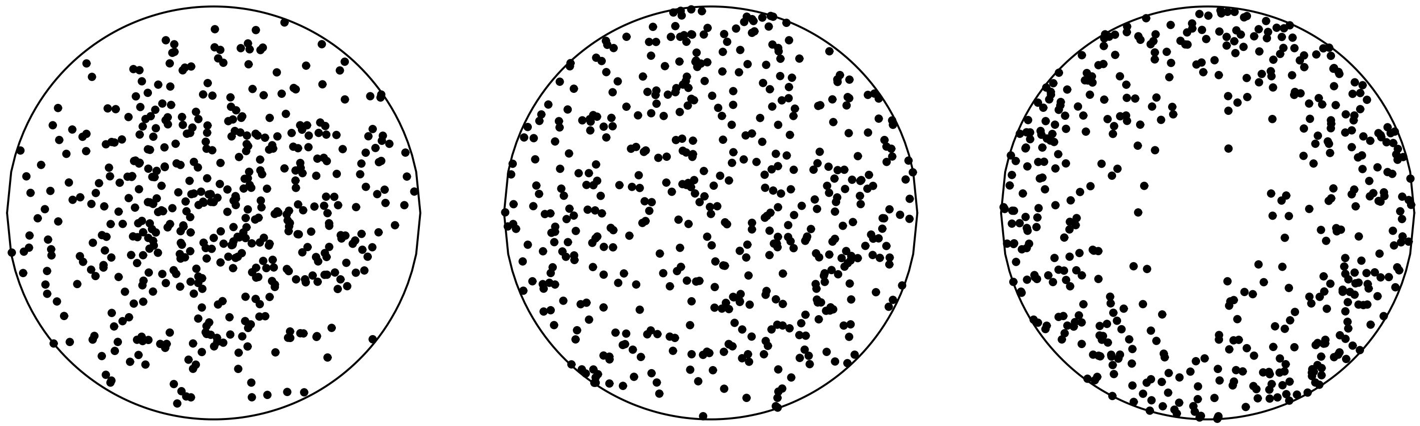
normal uniform LDI

if X is uniformly distributed on an n-ball



	normal	uniform	LDI
$\frac{1}{\lvert X \rvert}\sum_x V_r(x)$	0·1874	0·1247	0·0582

if X is uniformly distributed on an n-ball



(a) (b) (c)

	normal	uniform	LDI
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$$E[V_r(x)] = \frac{1}{2^{n+1}} = \frac{1}{2^3} = 0.125$$

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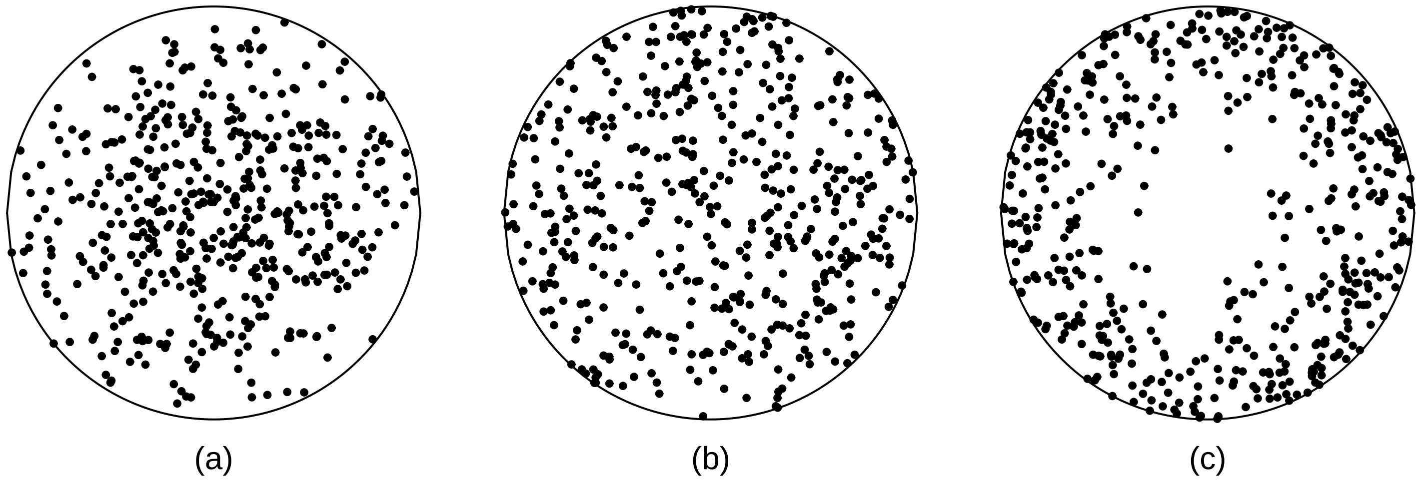

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even for small $|X|$, $\frac{1}{|X|}\sum_x V_r(x) \approx E[V_r(x)] = \frac{1}{2^{n+1}}$, for uniformly distributed X