Axiomatic Method & Rigor

Axiomatic Thinking
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I. Introduction

Axiomatic Thinking as Thinking with Awareness

"To proceed axiomatically means nothing other than to think with awareness (mit Bewußtsein denken)"

Hilbert, "Neubegründung der Mathematik. Erste Mitteilung" (1922), 161

This description of axiomatic thinking will be my preoccupation today.

Awareness, transparency and rigor

I'll consider it, in particular, as expressing a concern for rigor.

More specifically, I'll consider it as affirming an ideal of **transparency**, something which has generally been taken to be vital to the attainment of rigor.

More accurately, I'll consider it as an affirmation of one ideal of transparency and an associated ideal of rigor.

There were other ideals of rigor that had currency in the late nineteenth century, and I'll contrast Hilbert's with an important one of these—namely, Pasch's.

Thinking with Awareness: Pre-axiomatic vs. axiomatic reasoning

Concern for transparency on Hilbert's part was also indicated by descriptions in his early foundational writings of a general type of reasoning in mathematics—what I will call pre-axiomatic reasoning—that Hilbert found wanting in certain respects.

He seemed sometimes to emphasize features of pre-axiomatic reasoning that seem inimical to rigor, or at least to transparency based variants of it.

Consider, for example, the following description of (at least one type of) pre-axiomatic reasoning in arithmetic . . .

Pre-axiomatic reasoning in arithmetic

"[W]e do not always follow chains of thought-operations (Denkoperationen) in arithmetic back to the axioms, any more than we do in geometrical reflections. On the contrary we apply, especially in the first attack (ersten Inangriffnahme) on a problem, a rapid (rasches), unconscious (unbewußtes), not definitely secure (nicht definitiv sicheres) combination [of thought-operations, MD], trusting to a certain arithmetical feel (arithmetisches Gefühl) for the ways of operating with arithmetical signs (Wirkungsweise der arithmetischen Zeichen), which we could as little dispense with in arithmetic as we could dispense with the geometrical imagination (geometrische Einbildungskraft) in geometry."

Hilbert, "Mathematical Problems" (1900), 260

Pre-axiomatic reasoning & its obscuring features

In this remark, Hilbert portrays pre-axiomatic reasoning in arithmetic as having the following features:

It is

- rapid (rasches),
- unconscious (unbewußtes), and
- based on a "feel" (Gefühl) we have for operating with arithmetical signs.

Pre-axiomatic vs. axiomatic thinking

These features of pre-axiomatic reasoning seem to be contra-indicative of what I've been roughly referring to as the "transparency" of a piece of pre-axiomatic reasoning.

To proceed pre-axiomatically, then, at least on the understanding of that term I'm highlighting here, does not seem generally to have been to think with awareness of the various elements which together make up a—possibly intricate—given piece of pre-axiomatic reasoning.

It was, indeed, a major part of the task of axiomatic arithmetic to make explicit the various elements of what, logically speaking, our long-standing, deeply ingrained habits of pre-axiomatic arithmetical reasoning generally do not.

Historical aside: A parallel with Dedekind 1

The lack of transparency that Hilbert believed is characteristic of pre-axiomatic arithmetical reasoning is close in spirit to Dedekind's view of pre-axiomatic arithmetical reasoning too.

'[F]rom the time of birth we are continually and in increasing measure led to relate things to things and thus to exercise that faculty of the mind on which the creation of numbers depends; this exercise goes on continually . . . in our earliest years; the accompanying formation of judgments and chains of reasoning leads us to a store of real arithmetical truths which our first teachers later refer to as something simple, self-evident, and given in inner intuition. **Thus it happens that many very complicated notions . . . are erroneously regarded as something simple.**"

Dedekind, *Was sind und was sollen die Zahlen?* (1888), preface

Historical aside: A parallel with Dedekind 2

"I like to compare this action of thought [the formation of judgements and chains of reasoning in pre-axiomatic arithmetical thinking, MD], so difficult to trace on account of the rapidity of its performance, with the action which an accomplished reader performs in reading; this reading always remains a more or less complete repetition of the individual steps which the beginner has to take in his wearisome spelling-out; a very small part of the same, and therefore a very small effort or exertion of the mind, is sufficient for the *practiced* reader to recognize the correct, true word ..."

Dedekind, *Was sind und was sollen die Zahlen?* (1888), preface

Pre-axiomatic reasoning & its vindication

Pre-axiomatic reasoning thus presented a problem.

On the one hand, it was useful, perhaps even indispensable, as a practical means of advancing our knowledge.

On the other hand, its lack of transparency jeopardized its rigor and security . . . hence its standing as a genuine science.

There was thus reason to in some sense "retain" rather than displace it, but also reason to require its further elaboration and examination and, in light of this, its ultimate justification (albeit, perhaps, in a modified form).

Something like this, I believe, is what Hilbert hoped to achieve by his program for the foundation of arithmetic.

Hilbert and Pasch

Today, I want to focus on what I've been calling "transparency", and Hilbert's (possibly changing) understanding of it.

More exactly, I want to focus on what I see as some significant differences between the later views of Hilbert and the views of his contemporary, Pasch, whose views concerning rigor and the conditions of its attainment seem to have been influential.

I'll thus begin with a basic overview of Pasch's ideas concerning rigor and (at least a type of) transparency.

II. Pasch on rigor and transparency

Pasch on rigor and abstraction from meaning

In his *Vorlesungen über neuere Geometrie* (1882), Pasch made a well-known statement concerning the conditions of attaining rigor in geometrical proof.

The criterion he offered, which I will call the **abstractionist standard**, essentially called not only for the elimination of appeals to geometrical figures in geometrical proof, but for the elimination of all appeals to semantical contents of geometrical terms whatsoever.

Transparency as gaplessness

"[I]f geometry is to be genuinely deductive, the process of inferring (Process des Folgerns) must be everywhere independent of (unabhängig sein vom) the *sense* (*Sinn*) of geometrical concepts just as it must be independent of figures. It is only *relations* between geometrical concepts that should be taken into account in the propositions and definitions that are dealt with. In the course of a deduction, it is certainly permissible (statthaft) and useful (nützlich), though *by no means necessary* (*keineswegs nöthig*), to think of the meaning (Bedeutung) of the geometrical concepts involved. In fact, if it is necessary to so think, the gappiness (Lückenhaftigkeit) of the deduction and the insufficiency (Unzulänglichkeit) of the means of proof is thereby revealed . . ."

Vorlesungen über neuere Geometrie (1882), 98 (emphases (italics) in text)

Pasch's criterion for gaplessness

The part of this statement that most concerns me is that where Pasch states that the "process of inferring" in proper geometrical proof ought rightly to be "independent of" the meanings of geometrical terms.

I'll offer a view of what such independence might have amounted to for Pasch, and consider what he and others seem have taken to be at stake in making it a standard of proper geometrical reasoning.

Independence via abstraction from meaning

Pasch's call for "independence" seems to have been a call for restriction on the evidence that may rightly be used to justify judgments of deductive validity . . . the implication being that such judgements play an important part in geometrical proof, and, with the appropriate subjectival adjustments, in mathematical proof generally.

To get a capsule formulation of Pasch's condition, let $\mathcal{C}$ represent the conclusion of an inference and $\mathcal{P}$ represent its premises.

We can then formulate his envisioned abstraction condition for geometrical reasoning as follows:

Abstraction Condition: The justification of a judgment that an inference from $\mathcal{P}$ to $\mathcal{C}$ is deductively valid ought not to be based on any judgment whose contents concern the meanings of geometrical terms that occur in $\mathcal{P}$ and/or in $\mathcal{C}$.

Abstraction & Rigor

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Pasch held adherence to some such condition to be necessary for the attainment of deductive rigor in geometrical proof.

If this is right, then rigor and failure of rigor is fundamentally a matter of the evidence a reasoner uses in justifying judgements of deductive validity or implication . . . the implication being that such evidence is typically part of what figures in something's being a proof.

The nature of abstraction

Abstraction Condition: The justification of a judgment that an inference from $\mathcal{P}$ to $\mathcal{C}$ is deductively valid ought not to be based on any judgment whose contents concern the meanings of geometrical terms that occur in $\mathcal{P}$ and/or $\mathcal{C}$.

The 'abstraction' that figures here seems typically to have been conceived as a type of **prescission** (aka 'precision').

[Terminological Note: OED: "n. **1.** Chiefly *Philos.* The action or an act of separating or cutting off, esp. the mental separation of one fact or idea from another; abstraction, definition."]

Rigor & Premisory Surreption

Pasch did not therefore see failure of rigor as simply misjudgment of deductive validity.

Rather, he saw it as misjudgment of validity of a particular type—namely, that owing to unrecognized reliance on the contents of geometrical terms in the justification of judgements of deductive implication.

I'll focus, and I believe Pasch to have focused, mainly on **premisory surreption** or, rather, on vicious premisory surreption (*Vitium subreptionis*) as it has sometimes been called (cf. J. H. Lambert, "Theory der Parallelinien I" (1786), 156).

Premisory Surreption

By this is generally meant misjudgement of the validity of an inference due to misidentification of what, properly speaking, are its premises.

More particularly, it is misidentification of premises that consists in a reasoner's treating an inference as containing a premise which, properly speaking, it does not contain.

Premisory surreption & misidentification of premises

Seen this way, the deficiency represented by premisory surreption is not so much misjudgement of validity *per se* as misidentification of premises.

A judgement of validity based on premisory surreption might well be correct in the sense that inference which includes the surreptiously added premise(s) actually is valid.

What is fundamentally wrong, then, with judging a surreptious argument to be valid is not that, taken to include its surreptious premises, it is not valid, but that premises sufficient to make it valid have not been properly identified or registered as premises.

Premisory surreption: Key elements

Generally speaking, then, for a given inference *Inf*, and a given inferring agent *R*, premisory surreption by *R* concerning *Inf* fundamentally involves:

(i) Judgment by *R* that *Inf* is valid.

(ii) Failure by *R* to recognize that this judgment is based on her taking *Inf* to include a premise(s) which, for purposes of judging the validity of *Inf*, it can not properly be taken to include.

Inference and association

Pasch's proposed measure for guarding against premisory surreption was enforcement of the Abstraction Condition.

Why should, or at least might such a measure be thought to be effective?

A pertinent and historically sensible idea is this:

- because inference, like other successional forms of thinking, is subject to influences of psychological association, where
- repeated association of one idea with another, one proposition with another, etc. increases the likelihood of co-application (e.g., of concepts being "thought" together, propositions being affirmed together, etc.), regardless of whether such co-application is logically warranted or the reasoner is aware of it.

Association & surreption

To the extent that this is so, unconscious association of propositions could be a significant source of premisory surreption.

Experiences, thinkings, affirmations, etc. have contents, and patterns of succession among such mental events may induce corresponding contentual associations.

Generally speaking, such associational tendencies or practices would seem to expose reasoners to premisory surreption in proof.

One way to understand Pasch's advocacy of the Abstraction Condition, then, is as an attempt to limit the extent to which geometrical reasoning is subject to the surreptive threats of contentual association.

Inference & association

On my reading of him, then, Pasch proposed a standard of rigor which called for judgments of inferential validity to be "independent" not only of uses of diagrams, but of all justificative uses of semantical contents of geometrical terms in arriving at judgements of validity.

In doing so, he suggested that practical achievement of rigor requires a kind of psychological discipline in geometrical proof—a discipline which, if it is successful, psychologically separates the inferences in geometrical proofs from consideration of the contents of geometrical terms.

He also seems to have believed that the exercise of such psychological discipline is a practically effective means of at least mitigating the rigor-compromising risks of contentual association.

III. From Pasch to Hilbert

Paschian Transparency

Pasch's abstraction condition can be seen as aimed at achieving a type of transparency in geometrical reasoning—namely, transparency which consists in the exposure of surreptious premises in traditional geometrical thinking.

This, of course, was seen by Pasch as applying not only to pre-axiomatic reasoning in geometry but also, indeed, especially, to the inadequate axiomatic practices that had preceded him.

Since, however, the exposure of surreptious premises was of a piece with the exposure of deductive gaps in traditional geometrical reasoning Paschian abstraction can also be seen as having been aimed at making deductive reasoning more transparent by illuminating or at least exposing its logical gaps (Lücke).

Paschian Rigor: No logical gaps

The thinking seems generally to have been along the following lines:

- No Logical Gaps Thesis: Logical or even deductive† gaps are bad things in mathematical reasoning, and proper proofs do not contain them.

Strictly speaking, I believe, Hilbert rejected the No Logical Gaps Thesis.

Indeed, to have done so would seem to be a necessary consequence of his acceptance of the pivotal real vs. ideal distinction of his later foundational thinking.

†: I take a deductive gap to occur when, even though a proposition $\mathcal{P}$ may logically imply a proposition $\mathcal{C}$, this becomes evident only with further reticulation of the connection between $\mathcal{P}$ and $\mathcal{C}$.

Weyl's summation

Weyl presented the salient novelty of Hilbert's viewpoint point this way in commenting on Hilbert's criticisms of Brouwer:

"Before Hilbert constructed his proof theory everyone thought of mathematics as a system of contentual (inhaltliche), meaningful (sinnerfüllte), and evident (einsichtige) truths; this point of view was the common platform of all discussions. . . . Brouwer, like everyone else, required of mathematics that its theorems be (in Hilbert's terminology) "real propositions", meaningful truths."

"Diskussionsbemerkungen zu dem zweiten Hilbertschen Vortrag über die Grundlagen der Mathematik", *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* 6 (1928), 22

Hilbert's View

On the view Hilbert developed, mathematical reasoning—mathematical proof—did not generally require the logical connection of propositions.

Instead, it was at least often, perhaps even characteristically, a matter of carrying out formal thought processes (formaler Denkprozesse) according to certain rules.

What Hilbert termed "ideal propositions" (e.g., what he called the **formula** '$a = b = b + a$', as distinct from the **proposition** (communicated by) '$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$') were elements of formal thought processes, e.g., that by which the formula '$2 + 3 = 3 + 2$' is obtained from '$a + b = b + a$' by a certain scheme of syntactical substitutions.

A strong statement

"[I]n mathematics the objects of our thinking are concrete signs (konkreten Zeichen) themselves, whose shapes (Gestalt), according to the conception adopted, are immediately clear and re-cognizable (unmittelbar deutlich und wiedererkennbar). . . . The propositions (Aussagen) which constitute mathematics are replaced (umgesetzed) by formulae, so that, mathematics proper (die eigentliche Mathematik), becomes a stock of formulae (Bestande an Formeln). . . . A proof becomes an array of formulas given as such to our perceptual intuition.

[I]n my theory, contentual inference (inhaltliche Schließen) is replaced by outward manipulation of signs according to rules (äußeres Handeln nach Regeln). In this way the axiomatic method attains that reliability (Sicherheit) and perfection that it can and must reach if it is to become the basic instrument of all theoretical research."

Hilbert, "Die Grundlagen der Mathematik" (1928), 2, 4

Ideal proof & traditional proof

In Hilbert's view, so-called ideal propositions or formulae are often and importantly used in (what ought to be counted as) mathematical proofs.

This brings us to a question, though, as to how to think about transparency and rigor in the case of such proofs.

On the traditional view, a proof is a finite sequence of judgments (or other propositional attitude-takings) whose propositional contents are logically related in certain ways.

So conceived, a proof can be rigorous according to Pasch's standard because it makes sense to consider whether what has been used to justify the judgement of validity made any use of the contents of geometrical terms.

Ideal proofs & rigor

Ideal proofs are not like this.

Not all their constituent elements—specifically, not all ideal propositions they may make use of—have propositional contents or are even in principle capable of having them.

As a result, they are not all capable of entering into the type of logical relationship that exists between two propositions when one of them logically implies the other.

This being so, it would seem, ideal proofs can not strictly be rigorous (or fail in rigor, for that matter).

They could not therefore be coherently required to be so.

A question

What becomes of rigor, then, when the traditional contentual view of proof is replaced by a view of the type Hilbert seems to have advocated?

Is there room for a meaningful conception of rigor that might serve as an ideal of formal reasoning in something like the way that avoidance of premisory surreption serves as a general ideal of contentual mathematical proof?

I will now conclude by briefly indicating a few ideas for a possible positive response to this question.

IV. Rigor reconceived

Rigor and explicitness

The differences between genuinely logical reasoning and Hilbert's ideal reasoning are considerable, and they dictate a change in the conception of rigor.

On the conception I have in mind, the aim of rigor will no longer be avoidance of premisory surreption and other validity-nullifying gaps in reasoning.

Rather, it will be avoidance of deficiencies of explicitness in our reasoning.

Explicitness and exhibition

In Hilbert's view, as I see it, axiomatic reasoning was intended to avoid just such deficiencies.

Axiomatic proofs, or, more accurately, what Hilbert referred to as the proofs of *formaler Axiomatik*, were taken to be concrete objects that are distinguished from one another, and from non-proofs, by outwardly manifest—or exhibitable—characteristics.

It may be this type of explicitness or transparency—the explicitness or transparency of the exhibitable or displayable—that Hilbert intended to emphasize when, as in the epigraph at the beginning of this paper, he described axiomatic thinking as thinking with consciousness.

Concluding problem

Proper rigor should guarantee avoidance of logical surreption when reasoning is of such a type as is composed of genuine logical inferences.

In Hilbert's view, however, not all mathematical reasoning—not even all of what ought rightly to be counted as mathematical proof—is of this type, however.

We are therefore faced with the problem of whether and in what form rigor might reasonably be retained as an ideal of mathematics once mathematics is conceived, as Hilbert seems to have conceived it, as admitting ideal as well as real proofs.

Does rigor, as Pasch and others understood it, simply disappear as a general ideal for proofs?

Or is there a new and perhaps deeper understanding of what rigor is that may serve as a unified conception for both real and ideal proof?

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A note on exhibition

Roughly speaking, exhibition in the current sense consists in the presentation (whatever, exactly, that might mean) of a particular concrete expression as an exemplar for other concrete expressions—specifically, expressions whose external features are sufficiently similar to those of the exemplar to qualify them as tokens of the same type as it.

Historical digression: Pasch's priority as an advocate of abstraction

It has been suggested that Pasch had some type of priority as a defender of the Abstractionist Standard of rigor.

Hans Freudenthal, for example, referred to him as "the father of rigor in geometry" (cf. "The main trends in the foundations of geometry in the 19th century" (1962), 619).

And some fifty years before Freudenthal's statement, J. W. Young remarked Pasch's priority as an advocate of a view of mathematical reasoning that seems to have seen it as being properly controlled by something like an abstraction condition.

"The abstract formulation of mathematics", he wrote, "seems to date back to the German mathematician Moritz Pasch." (cf. *Lectures on Fundamental Concepts of Algebra and Geometry* (1911), 51).

Historical digression: Lambert as an advocate of abstraction

Later, Young noted the link that Pasch asserted between abstraction and rigor—namely, that "to be rigorous ... an argument must be abstract" (op. cit., 218) in something like the current sense.

There were, however, clear expressions of similar abstractionist ideas before Pasch, most significantly, perhaps, in the writings of J. H. Lambert ...

Historical digression: Lambert on the abstraction requirement in geometrical proof

"[It] can and must it be required that one nowhere in a proof call on the thing itself (auf die Sache selbst berufe) but that the proof should be carried forward symbolically throughout (durchaus symbolisch vortrage)—if this is possible. In this aspect Euclid's postulates are the same as so many algebraic equations which one has before oneself, and from which x, y, z &c will be brought out (herausgebracht) without one's looking back at the thing itself (ohne daß man auf die Sache selbst zurücke sehe)."

"Theory der Parallelinien I", *Leipziger Magazin für reine und angewandte Mathematik* 1 (2) (1786), 149–150

Historical digression: Lambert on the independence of the parallel postulate

Lambert held, in particular, that proper derivation of the parallel postulate from the other basic Euclidean propositions would require derivation which

- "abstracts" (abstrahiert) (loc. cit.) from all "representation and conceivability of the things talked about" (von der Vorstellung und der Gedenkbarkeit der Sache die Rede ist) ("Theory der Parallelinien I" (1786), 155),

and which thus

- proceeds by the application of what are essentially symbolical rules.

Lambert on symbolic method as an antidote for surreption

Only in this way, Lambert suggested, can there be adequate protection against surreptitious importation of information (ein *Vitium subreptionis*, op. cit., 156) into—hence failure of rigor of—proofs of the parallel postulate from other principles of Euclidean geometry.