On the performance of axiom systems

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Axioms form the substantial part of a mathematical proof.

The aim of this lecture is to illustrate that axiom systems carry characteristic ordinals which serve as a measure for the performance of the axiom system.

Let $\mathfrak{M}$ be an abstract structure. There are numerous ordinals (GRT–ordinals) which in Generalized Recursion Theory are regarded as chacteristic for the structure. Examples:

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$\delta^{\mathfrak{M}}$, ($\sigma^i_n(\mathfrak{M})$, $\pi^i_n(\mathfrak{M})$, $\delta^i_n(\mathfrak{M})$) which are the suprema of the ordertypes of well–orderings which are definable (by a Σ^i_n, Π^i_n, Δ^i_n formula, respectively,) in the language of $\mathfrak{M}$.

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$\kappa^{\mathfrak{M}}$, $(\kappa^{\mathfrak{M}}_{\Sigma^i_n},\ \kappa^{\mathfrak{M}}_{\Pi^0_n},\ \kappa^{\mathfrak{M}}_{\Delta^0_n})$ which are the suprema of the closure ordinals of operators which are positively definable (by a $\Sigma^0_n, \Pi^0_n, \Delta^0_n$ formula, respectively) in $\mathcal{L}(\mathfrak{M})$.

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Given an axiomatization T for the structure $\mathfrak{M}$ there are obvious modifications for the ordinals $\delta^0_n(\mathfrak{M})$, $\sigma^0_n(\mathfrak{M})$, ...

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The ordinal $\delta^{\mathfrak{M}}_n(\mathsf{T})$ is the supremum of the ordertypes of orderings which are Δ^0_n definable in the language of $\mathfrak{M}$ such that T proves their well–foundedness.

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The distance between $\delta^{\mathfrak{M}}(\mathsf{T})$ and $\delta^{\mathfrak{M}}$ is a measure for the performance of an axiom system T.

Well–foundedness of an order relation $\prec$ is expressed by the Π^1_1–sentence

$$(\forall X)[(\forall x)[(\forall y)[y \prec x \to y \in X] \to x \in X] \to \mathit{field}(\prec) \subseteq X]$$

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To stay within the elementary language $\mathcal{L}(\mathfrak{M})$ we express well–foundedness by the pseudo Π^1_1–sentence (p-Π^1_1–sentence)

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Semantically we treat p-Π^1_1–sentences as full second order Π^1_1–sentences, i.e.

$$\mathfrak{M} \models F(X) \;:\Leftrightarrow\; (\mathfrak{M}, S) \models F[S] \text{ for all sets } S \subseteq \mathfrak{M}.$$

Therefore we have the formal definition

$$\delta^{\mathfrak{M}}(\mathsf{T}) := \sup\{\mathrm{otyp}(\prec) \mid \mathsf{T} \vdash \mathit{Prog}(\prec, X) \to \mathit{field}(\prec) \subseteq X\}$$

for order relations that are definable in the language of $\mathfrak{M}$.

For countable structures $\mathfrak{M}$ the link between its GRT–ordinals and their prooftheoretic counterparts is given by the notion of the Π^1_1–ordinal of $\mathfrak{M}$, which in turn is defined via a semi–formal system.

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An $\mathcal{L}(\mathfrak{M})$–sentence belongs to $\bigwedge$–type if all the members of $\mathrm{CS}(F)$ are needed to verify F and to $\bigvee$–type if some of the members of $\mathrm{CS}(F)$ suffice.

As an example let $\mathfrak{M}$ be a countable structure and $\mathcal{L}(\mathfrak{M})$ its elementary first order language.

Definition

The $\bigwedge$–type of $\mathcal{L}(\mathfrak{M})$ comprises

- the diagram of $\mathfrak{M}$,
- all sentences of the form $F \land G$ and $(\forall x)F(x)$.

The $\bigvee$–type of $\mathcal{L}(\mathfrak{M})$ comprises

- all false atomic sentences of $\mathfrak{M}$,
- all formulae of the form $F \lor G$ and $(\exists x)F(x)$.

Definition (The decoration of $\mathcal{L}(\mathfrak{M})$–sentences)

- $\mathrm{CS}(F) = \emptyset$ for atomic sentences F
- $\mathrm{CS}(F \circ G) = \langle F, G \rangle$ for $\circ \in \{\wedge, \vee\}$
- $\mathrm{CS}((\mathsf{Q}x)F(x)) = \big\langle F_x(\underline{m}) \big|\ m \in |\mathfrak{M}| \big\rangle$ for $\mathsf{Q} \in \{\forall, \exists\}$.

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Definition (The verification tree $\mathfrak{M} \models^{\alpha} F$)

($\bigwedge$) If $\mathfrak{M} \models^{\alpha_\iota} G_\iota$ and $\alpha_\iota < \alpha$ for all $G_\iota \in \mathrm{CS}(F)$ then $\mathfrak{M} \models^{\alpha} F$ holds true.

($\bigvee$) If $\mathfrak{M} \models^{\alpha_0} G_\iota$ and $\alpha_0 < \alpha$ for some $G_\iota \in \mathrm{CS}(F)$ then $\mathfrak{M} \models^{\alpha} F$ holds true.

Definition (The decoration of p-Π^1_1–$\mathcal{L}(\mathfrak{M})$–sentences)

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Pseudo Π^1_1 sentences of the form $s \in X$ cannot be verified. However, verifiable are formulae $s \in X \vee s \notin X$. Therefore we extend the verification calculus to a semi–formal proof relation $\mathfrak{M} \,\big|\!\frac{\alpha}{\rho}\, \Delta$ for finite sets Δ of p-Π^1_1–sentences which are to be interpreted as finite disjunction.

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Definition (The semi–formal system $\mathfrak{M} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta$)

($\bigwedge$) If $\mathfrak{M} \overset{\alpha_\iota}{\underset{\rho}{\vdash}} \Delta, G_\iota$ and $\alpha_\iota < \alpha$ for all $G_\iota \in \mathrm{CS}(F)$ then $\mathfrak{M} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, F$ holds true.

($\bigvee$) If $\mathfrak{M} \overset{\alpha_0}{\models} G_\iota$ and $\alpha_0 < \alpha$ for some $G_\iota \in \mathrm{CS}(F)$ then $\mathfrak{M} \overset{\alpha}{\models} F$ holds true.

Definition (The semi–formal system $\mathfrak{M} \models^{\alpha}_{\rho} \Delta$)

($\bigwedge$) If $\mathfrak{M} \models^{\alpha_\iota}_{\rho} \Delta, G_\iota$ and $\alpha_\iota < \alpha$ for all $G_\iota \in \mathrm{CS}(F)$ then $\mathfrak{M} \models^{\alpha}_{\rho} \Delta, F$ holds true.

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(X) If $\mathfrak{M} \models s = t$ then $\mathfrak{M} \,\big|\!\frac{\alpha}{\rho}\, \Delta, s \notin X, t \in X$ holds true for all ordinals α and ρ.

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(X) If $\mathfrak{M} \models s = t$ then $\mathfrak{M} \,|\frac{\alpha}{\rho}\, \Delta, s \notin X, t \in X$ holds true for all ordinals α and ρ.

(cut) If $\mathfrak{M} \,|\frac{\xi}{\rho}\, \Delta, F$ and $\mathfrak{M} \,|\frac{\xi}{\rho}\, \Delta, \neg F$ and $\mathrm{rnk}(F) < \rho$ then $\mathfrak{M} \,|\frac{\alpha}{\rho}\, \Delta$ for all ordinals $\alpha > \xi$.

Observations

(a) *For an* $\mathcal{L}(\mathfrak{M})$ *sentence* F *we have* $\mathfrak{M} \models^{\alpha} F$ *iff* $\mathfrak{M} \vdash^{\alpha}_{0} F$

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(c) $\mathfrak{M} \mid\!\frac{\alpha}{\rho}\, \Delta$, $\alpha \leq \beta$, $\rho \leq \sigma$ *and* $\Delta \subseteq \Gamma$ *imply* $\mathfrak{M} \mid\!\frac{\beta}{\sigma}\, \Gamma$.

Theorem (Π^1_1–completeness)

Let $\mathfrak{M}$ be a countable structure. Then $\mathfrak{M} \models (\forall X)F(X)$ iff there is a countable ordinal α such that $\mathfrak{M} \,\big|\!\frac{\alpha}{0}\, F(X)$.

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Definition (The Π^1_1–ordinal of a countable structure)

For a p-Π^1_1 sentence F define

$$\mathrm{tc}(F) := \begin{cases} \min\{\alpha \mid \mathfrak{M} \mid\frac{\alpha}{0} F\} & \text{if such an } \alpha \text{ exists} \\ \omega_1 & \text{otherwise} \end{cases}$$

and

$$\pi^{\mathfrak{M}} := \sup\{\mathrm{tc}(F) \mid \mathfrak{M} \models F\}.$$

Lemma (Boundedness Lemma)

For a countable structure and an order relation $\prec$ that is definable in $\mathcal{L}(\mathfrak{M})$ and a finite set of X–positive p-Π^1_1–sentences $\Delta(X)$ we have

$$\mathfrak{M} \mid\!\frac{\alpha}{0}\, \neg Prog(\prec, X), \Delta(X) \;\Rightarrow\; \mathfrak{M} \models \Delta(X)[\prec\restriction\alpha].$$

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Corollary

For a countable structure $\mathfrak{M}$ we have $\delta^{\mathfrak{M}} \leq \pi^{\mathfrak{M}}$.

Definition

A structure $\mathfrak{M}$ is acceptable if it contains a copy $N^{\mathfrak{M}}$ of the natural numbers together with a coding machinery that is definable in $\mathcal{L}(\mathfrak{M})$.

Theorem

For an acceptable countable structure we have $\delta^{\mathfrak{M}} = \pi^{\mathfrak{M}}$.

Definition

Let T be an axiom system for a countable structure $\mathfrak{M}$. Then we put

$$\pi^{\mathfrak{M}}(\mathsf{T}) := \sup\left\{\mathrm{tc}(F) \mid \mathsf{T} \vdash F\right\}$$

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As a corollary to the Boundedness Theorem we have

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Theorem

Let T *be an axiom system for a countable structure* $\mathfrak{M}$ *and* T' *an extension of* T *by a true* Σ^1_1*–sentence. Then* $\delta^{\mathfrak{M}}(\mathsf{T}') \leq \pi^{\mathfrak{M}}(\mathsf{T})$.

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Proof Let $\mathsf{T}' = \mathsf{T} + (\exists Y)G(Y)$ and assume

$$\mathsf{T}' \vdash \mathit{Prog}(\prec, X) \to \mathit{field}(\prec) \subseteq X.$$

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So there is an ordinal $\alpha < \pi^{\mathfrak{M}}(\mathsf{T})$ such that

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Hence

$$\mathfrak{M} \models \neg G(Y) \vee \mathit{field}(\prec) \subseteq \prec\!\restriction\!\alpha$$

by the Boundedness Lemma. Since there is an $S \subseteq |\mathfrak{M}|$ such that $\mathfrak{M} \models G(S)$ this entails the claim.

Definition (Acceptable axiomatizations)

An axiom system T for a countable acceptable structure $\mathfrak{M}$ is acceptable if it proves all the properties of the coding machinery.

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Remark

The ordinal $\delta^{\mathfrak{M}}(\mathsf{T})$ of an acceptable axiom system is rather a measure for its performance in respect to a universe above $\mathfrak{M}$ than to $\mathfrak{M}$ itself. To improve the performance of T it thus has to be strengthened by axioms talking about an universe above $\mathfrak{M}$.

Definition (Veblen Functions)

The function φ_0 enumerates the additively indecomposable ordinals, i.e. $\varphi_0(\alpha) = \omega^\alpha$.
For $\xi > 0$ the functions φ_ξ enumerate the common fixed–points of the functions φ_ζ for $\zeta < \xi$.

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Theorem (Cut Elimination Theorem)

Let $\mathfrak{M}$ be a countable structure. Then $\mathfrak{M} \mathrel{\Big|\!\frac{\alpha}{\beta+\omega^\rho}} \Delta$ implies $\mathfrak{M} \mathrel{\Big|\!\frac{\varphi_\rho(\alpha)}{\beta}} \Delta$.

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Embed a formal proof $\mathsf{T} \vdash F$ into the semi–formal system to obtain ordinals α_F and ρ_F such that $\mathfrak{M} \,\big|\!\frac{\alpha_F}{\omega^{\rho_F}}\, F$.

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Use cut elimination to obtain $\mathfrak{M} \,\Big|\frac{\varphi_{\rho_F}(\alpha_F)}{0}\, F$.

Infer $\pi^{\mathfrak{M}}(\mathsf{T}) \leq \sup\{\varphi_{\rho_F}(\alpha_F) |\; F \in \mathcal{L}(\mathfrak{M})\}$ by the Boundedness Theorem.

Definition

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Theorem

Let T *be an Peano–like axiomatization of a countable structure* $\mathfrak{M}$. *Then* $\delta^{\mathfrak{M}}(\mathrm{T}) = \pi^{\mathfrak{M}}(\mathrm{T}) = \varphi_1(0) = \varepsilon_0$.

Remark

If $\mathfrak{M}$ is an acceptable countable structure whose sentences have all complexities below Γ_0 and T is an acceptable axiomatization of $\mathfrak{M}$ whose "universe axioms" can be verified with lengths below Γ_0 we obtain $\pi^{\mathfrak{M}}(\mathsf{T}) \leq \Gamma_0$.

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As a consequence we obtain that the well–foundedness of an order relation of ordertype Γ_0 cannot be proved by an axiom system T whose embedding yields $\mathfrak{M} \mathrel{|\!\frac{\alpha}{\rho}} \neg Prog(\prec, X), field(\prec) \subseteq X$ with ordinals $\alpha, \rho < \Gamma_0$.

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This shows that the ordinal Γ_0 cannot be justified "from below". On the other hand Kurt Schütte and Sol Feferman could show (independently) that every ordinal less than Γ_0 is justifiable from below. For this reason Γ_0 is regarded as the limiting ordinal of predicativity.

The largest "analytic" universe above $\mathfrak{M}$ is the structure $\mathbb{A}(\mathfrak{M}) = (\mathfrak{M}, \mathrm{Pow}(|\mathfrak{M}|))$, a structure which is not longer countable.

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The largest analytic universe which could be amenable to prooftheoretic studies is $\mathbb{A}^2(\mathfrak{M}) = (\mathfrak{M}, \mathrm{Pow}^2(\mathfrak{M}))$ where $\mathrm{Pow}^2(\mathfrak{M})$ contains the subsets of $|\mathfrak{M}|$ which are second order definable in $\mathfrak{M}$.

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An extension of an axiom system T for $\mathfrak{M}$ to an axiom system for $\mathbb{A}^2(\mathfrak{M})$ can be obtained by adding the comprehension scheme

$$(\mathrm{CA}) \qquad (\exists X)(\forall x)[x \in X \leftrightarrow F(x)],$$

where F is supposed to vary over all second oder formulae of $\mathcal{L}(\mathfrak{M})$ which do not contain the variable X freely.

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Call an axiom system analytic if it axiomatizes subuniverses of $\mathbb{A}^2(\mathfrak{M})$.

A well studied example for analytic axiom systems are iterated inductive definitions.

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For an X–positive formula $F(X,x)$ in the language $\mathcal{L}(\mathfrak{M})$ let

$$\Phi_F(S) := \{m \in |\mathfrak{M}| \mid \mathfrak{M} \models F(S,x)\}.$$

This defines a monontone operator

$$\Phi_F \colon \mathrm{Pow}(|\mathfrak{M}|) \longrightarrow \mathrm{Pow}(|\mathfrak{M}|),$$

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Iterating Φ_F from below defines the stages $\Phi_F^\alpha = \Phi_F(\Phi_F^{<\alpha})$.

Then there is a least ordinal σ such that $\Phi_F^\sigma = \Phi_F^{<\sigma} = I_F$ the closure ordinal $||\Phi_F||$ of Φ_F. For an element $n \in I_F$ let $|n|_F := \min\{\xi \mid n \in \Phi_F^\xi\}$ denote its F–norm.

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We then define

$$\kappa^{\mathfrak{M}} := \sup\{||\Phi_F|| \mid F \text{ is } X\text{–positve } \} = \sup\{|n|_F + 1 \mid F \text{ is } X\text{–positive } \wedge n \in I_F\}.$$

A subset $S \subseteq |\mathfrak{M}|$ is positive–inductively definable over $\mathfrak{M}$ if there is an $s \in |\mathfrak{M}|$ such that S is the s–slice $\{x \mid \langle x, s\rangle \in I_F\}$ for some X–positive formula F in $\mathcal{L}(\mathfrak{M})$.

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Let $\Gamma(\mathfrak{M})$ be the collection of all inductively definable subsets of $|\mathfrak{M}|$ and

$$\mathfrak{M}_0 := \mathfrak{M}, \quad \Gamma_1(\mathfrak{M}) := \Gamma(\mathfrak{M}_0),$$

$$\Gamma_{\mu+1}(\mathfrak{M}) := \Gamma_\mu(\mathfrak{M}) \cup \Gamma(\mathfrak{M}_\mu), \;\; \mathfrak{M}_{\mu+1} := (\mathfrak{M}, \Gamma_{\mu+1}(\mathfrak{M})),$$

$$\Gamma_{<\lambda}(\mathfrak{M}) := \{S \,|\; S = \{x \in |\mathfrak{M}| \,|\; (\exists \xi < \lambda)(\exists S_\xi \in \Gamma_\xi(\mathfrak{M}))[x \in S_\xi]\}\}$$

$\Gamma_\lambda(\mathfrak{M}) := \bigcup_{\xi<\lambda} \Gamma_\xi(\mathfrak{M}) \cup \Gamma_{<\lambda}(\mathfrak{M})$ and $\mathfrak{M}_\lambda := (\mathfrak{M}, \Gamma_\lambda(\mathfrak{M}))$ for limit ordinals λ.

Let $\kappa_0^{\mathfrak{M}} := 0$, $\kappa_{\mu+1}^{\mathfrak{M}} := \kappa^{\mathfrak{M}_\mu}$ and $\kappa_\lambda^{\mathfrak{M}} := \sup_{\xi<\lambda} \kappa_\xi^{\mathfrak{M}}$.

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Since

$$s \in I_F \;\Leftrightarrow\; \mathfrak{M} \models \underbrace{(\forall x)[F(X,x) \to x \in X] \to s \in X}_{I_F(X,s)}$$

we get by (a variant of) the Boundedness Theorem

$|s|_F \leq 2^{\mathrm{tc}\left(I_F(X,s)\right)}$, hence $\kappa^{\mathfrak{M}} \leq \pi^{\mathfrak{M}}$.

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$$|s|_F \leq 2^{\mathrm{tc}\left(I_F(X,s)\right)}, \text{ hence } \kappa^{\mathfrak{M}} \leq \pi^{\mathfrak{M}}.$$

For a countable acceptable structure $\mathfrak{M}$ we thus have

$\kappa^{\mathfrak{M}} \leq \pi^{\mathfrak{M}} = \delta^{\mathfrak{M}} \leq \kappa^{\mathfrak{M}}$.

Fixed–points of positively definable operators are easily axiomatized by their closure conditions.

(ID^1_μ) $\quad (\forall x)[F(I_F, x) \to x \in I_F],$

(ID^2_μ) $\quad (\forall x)[F(G, x) \to G(x)] \to I_F \subseteq \{x \mid G(x)\}$

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Let T be an axiom system for an acceptable countable structure $\mathfrak{M}$. By $ID_\nu(\mathrm{T})$ we understand T augmented by all schemes ID^k_μ for $k = 1, 2$ and $\mu < \nu$ where G in ID^2_μ varies over the full language of $\mathfrak{M}_\nu$.

Given an axiomatization T for the theory $\mathfrak{M}_\mu$ we define

$$\kappa^{\mathfrak{M}_\mu}(\mathsf{T}) := \sup\{|s|_F + 1 \mid \mathsf{T} \vdash s \in I_F\}$$

where $F(X,x)$ varies over the X–positive formulae in the language of $\mathfrak{M}_\mu$.

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For an acceptable axiomatization T for a countable acceptable structure $\mathfrak{M}$ and $\mu \leq \nu$ we then obtain

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Observe that in contrast to the ordinals $\pi^{\mathfrak{M}}(\mathsf{T})$ and $\delta^{\mathfrak{M}}(\mathsf{T})$—whose definitions need mandatorely p-Π^1_1–sentences—the definition of $\kappa^{\mathfrak{M}}(\mathsf{T})$ needs no free second order variables.

To compute bounds for the stages $|s|_F$ we need a finer grained structure $\mathfrak{M}^*_\nu$ which also contains constants $I_F^{<\xi}$ for the stages $\Phi_F^{<\xi}$. In the semi-formal systems for $\mathfrak{M}^*_\mu$ we can dispense with the (X)–rule but have to axiomatize the closure ordinals $\kappa^{\mathfrak{M}_\mu}$ for which we introduce "ideal" ordinals $\Omega_{\mu+1}$ and their defining rules

$(\Omega_{\mu+1})$ $\quad \mathfrak{M}^*_\nu \mid\!\frac{\alpha_0}{\rho}\, \Delta, F(I_F^{<\Omega_{\mu+1}}, s)$ and $\alpha_0 < \alpha$ for an X–positive formula $F(X,x)$ in $\mathrm{L}(\mathfrak{M}_\mu)$ imply $\mathfrak{M}^*_\nu \mid\!\frac{\alpha}{\rho}\, \Delta, s \in I_F^{<\Omega_{\mu+1}}$.

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Theorem (Boundedness for $\mathfrak{M}^*_\nu$)

If $\mathfrak{M}^*_\nu \,\Big|\!\frac{\alpha}{\rho}\, \Delta(I_F^{<\Omega_{\mu+1}})$ for $F \in \mathcal{L}(\mathfrak{M}_\mu)$ and $\alpha < \Omega_{\mu+1}$ then $\mathfrak{M}^*_\nu \,\Big|\!\frac{\alpha}{\rho}\, \Delta(I_F^{<\zeta})$ holds true for $\alpha \le \zeta \le \Omega_{\mu+1}$. Hence $\mathfrak{M}^*_\nu \,\Big|\!\frac{\alpha}{\rho}\, s \in I_F^{<\Omega_{\mu+1}}$ entails $|s|_F < \alpha$.

A translation from the language of $ID_\nu(\mathsf{T})$ into the language of $\mathfrak{M}^*_\nu$ is obtained by replacing constants I_F for $F \in \mathcal{L}(\mathfrak{M}_\mu)$ by constants $I_F^{<\Omega_{\mu+1}}$. Unravelling a formal proof $ID_\nu(\mathsf{T}) \vdash F$ into a semi-formal proof $\mathfrak{M}^*_\nu \,\Big|\!\frac{\alpha}{\rho}\, F$ will in general produce ordinals α that are far too large.

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Such a thinned set can be provided by α–iterated Skolem hull operators $\mathcal{H}^\alpha$.

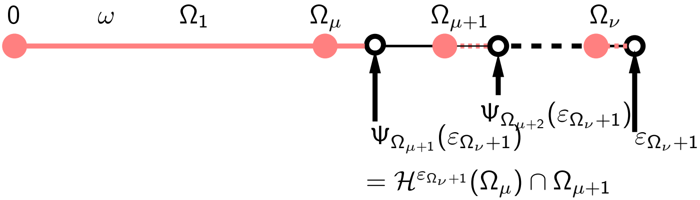

Figure: The ordinal set $\mathcal{H}^{\varepsilon_{\Omega_\nu+1}}(\Omega_\mu)$

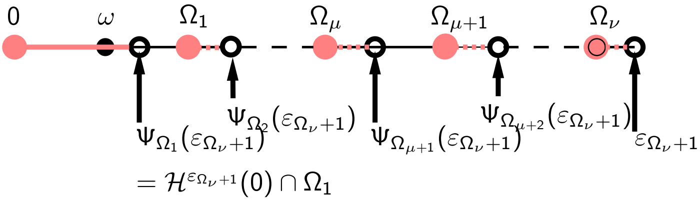

Figure: The ordinal set $\mathcal{H}^{\varepsilon_{\Omega_\nu+1}}(0)$

Let $\mathcal{L}(\mathfrak{M}^*_\mu)^+$ denote the sentences in $\mathcal{L}(\mathfrak{M}^*_\mu)$ which contain only occurrences of constants $I_F^{<\xi}$ for $\xi < \Omega_{\mu+1}$ and at most positive occurrences of $I_F^{<\Omega_{\mu+1}}$.

Theorem (Collapsing Theorem (roughly stated))

Let Δ be a set of $\mathcal{L}(\mathfrak{M}^*_\mu)^+$–sentences. Then $\mathfrak{M}^*_\nu \mathrel{\Big|\!\frac{\alpha}{\Omega_\nu}} \Delta$ implies

$$\mathfrak{M}^*_\nu \mathrel{\Big|\!\frac{\Psi^{\tilde{\alpha}}_{\Omega_{\mu+1}}}{\Psi^{\tilde{\alpha}}_{\Omega_{\mu+1}}}} \Delta\,.$$

The standard method of predicative proof theory is

Embed a formal proof $\mathrm{T} \vdash F$ into the semi–formal system to obtain ordinals α_F and ρ_F such that $\mathfrak{M} \left|\frac{\alpha_F}{\omega^{\rho_F}}\right. F$.

Use cut elimination to obtain $\mathfrak{M} \left|\frac{\varphi_{\rho_F}(\alpha_F)}{0}\right. F$.

Infer $\pi^{\mathfrak{M}}(\mathrm{T}) \leq \sup\{\varphi_{\rho_F}(\alpha_F) \mid F \in \mathcal{L}(\mathfrak{M})\}$ by the Boundedness Theorem.

The method has to be altered to

Embed a formal proof $ID_\nu(\mathsf{T}) \vdash s \in I_F$ for $F \in \mathcal{L}(\mathfrak{M}_\mu)$ into the semi–formal system $\mathfrak{M}^*_\nu$ to obtain ordinals $\alpha < \varepsilon_{\Omega_\nu+1}$ and $\rho < \Omega_\nu + \omega$ such that $\mathfrak{M}^*_\nu \mathrel{\Big|\!\!\frac{\alpha}{\rho}} s \in I_F^{<\Omega_{\mu+1}}$.

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Use cut elimination to obtain $\mathfrak{M}^*_\nu \, \vdash^{<\varepsilon_{\Omega_\nu+1}}_{\Omega_\nu} s \in I_F^{<\Omega_{\mu+1}}$ and collapsing to get $\mathfrak{M}^*_\nu \, \vdash^{<\Psi^{\varepsilon_{\Omega_\nu+1}}_{\Omega_{\mu+1}}}_{\Omega_{\mu+1}} s \in I_F^{<\Omega_{\mu+1}}$.

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Use boundedness to infer $|s|_F < \Psi^{\varepsilon_{\Omega_\nu+1}}_{\Omega_{\mu+1}}$ for all $F \in \mathcal{L}(\mathfrak{M}_\mu)$, hence

$$\kappa^{\mathfrak{M}_\mu}(\mathbf{ID}_\nu(\mathsf{T})) \leq \Psi^{\varepsilon_{\Omega_\nu+1}}_{\Omega_{\mu+1}}.$$

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Use cut elimination to obtain $\mathfrak{M}^*_\nu \mathrel{\Big|\!\frac{<\varepsilon_{\Omega_\nu+1}}{\Omega_\nu}} s \in I_F^{<\Omega_{\mu+1}}$ and collapsing to get $\mathfrak{M}^*_\nu \mathrel{\Big|\!\frac{<\Psi_{\Omega_{\mu+1}}^{\varepsilon_{\Omega_\nu+1}}}{\Omega_{\mu+1}}} s \in I_F^{<\Omega_{\mu+1}}$.

Use boundedness to infer $|s|_F < \Psi_{\Omega_{\mu+1}}^{\varepsilon_{\Omega_\nu+1}}$ for all $F \in \mathcal{L}(\mathfrak{M}_\mu)$, hence $\delta^{\mathfrak{M}_\mu}(\mathbf{ID}_\nu(\mathsf{T})) = \kappa^{\mathfrak{M}_\mu}(\mathbf{ID}_\nu(\mathsf{T})) \leq \Psi_{\Omega_{\mu+1}}^{\varepsilon_{\Omega_\nu+1}}$.

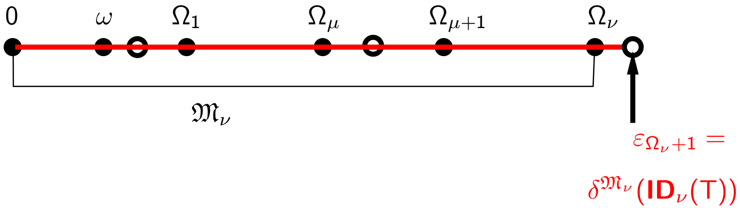

Figure: $\delta^{\mathfrak{M}_\nu}(\mathbf{ID}_\nu(\mathsf{T})) = \mathcal{H}^{\varepsilon_{\Omega_\nu+1}}(\Omega_\nu)$

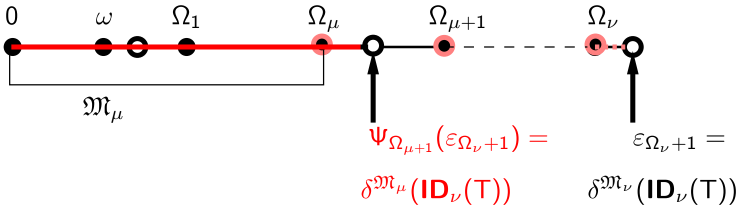

Figure: $Spec^{\mathfrak{M}_\mu}(\mathbf{ID}_\nu(\mathsf{T})) = \mathcal{H}^{\varepsilon_{\Omega_\nu+1}}(\Omega_\mu)$

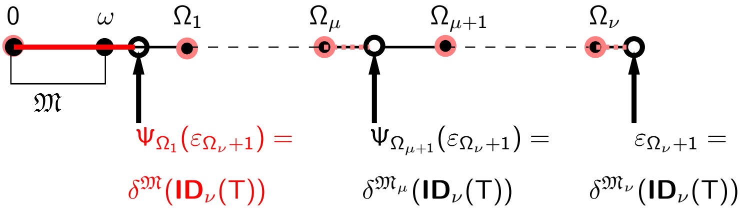

Figure: $Spec^{\mathfrak{M}}(\mathbf{ID}_\nu(\mathsf{T})) = \mathcal{H}^{\varepsilon_{\Omega_\nu+1}}(0)$

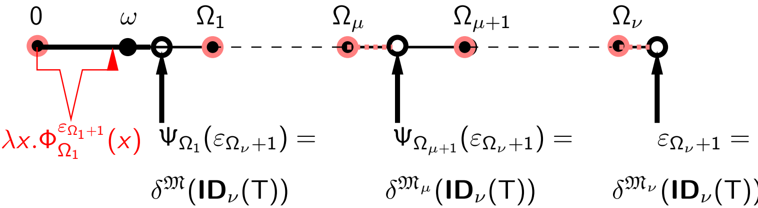

Figure: $\mathit{Spec}^{\mathfrak{M}}(\mathbf{ID}_\nu(\mathsf{T}))$ extended

Axioms for set theoretical universes above $\mathfrak{M}$ can be treated by similar methods. The analyses of iterations of Kripke Platek axiom systems follow the same pattern and are even simpler to handle (theething troubles causes—as in forcing techniques—extensionality).

Substantial gain in performance is obtained by axiomatizing reflection principles. The strongest systems which can be (at least partly) handled today are Kripke Platek set theories with stability axioms.

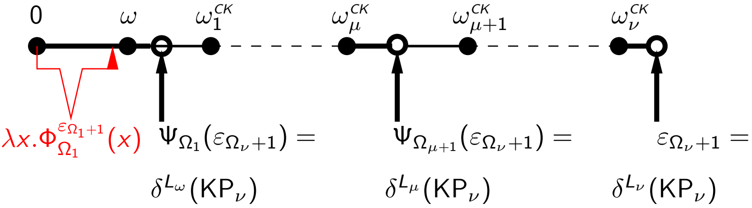

Figure: $Spec^{\mathfrak{M}}(\mathsf{KP}_\nu)$ extended

Thank you for your attention