Predicativity and parametric polymorphism of Brouwerian implication
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1. Two well-known examples of impredicative definitions are
   a) k = the largest number in $\{1, 2, 3\}$; k is a member of that totality.
   b) Let s be an infinite set. According to the power set axiom, there exists a set $\mathcal{P}(s) = \imath x(\forall y(y \in x \leftrightarrow y \subseteq s))$. Here $\forall y$ is a quantification over the totality of all sets; $\mathcal{P}(s)$ is a member of that totality.

2. 'A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.'
   [Troelstra and van Dalen (1988), *Constructivism in Mathematics*, vol.1, p.9]

3. a proof of A that combines subproofs of $A \rightarrow B$ and $(A \rightarrow B) \rightarrow A$

4. f is a function such that, for any x in the totality of all intuitionistic proofs, if x is a proof of A, then $f(x)$ is a proof of B.

5. From the assumption that A is true, a proof of B can be obtained.

6. 'truth is only in reality i.e. in the present and past experiences of consciousness'
   [Brouwer (1949), 'Consciousness, Philosophy, and Mathematics', p.1243]

7. 'Therefore you may be doubtfull [sic] as to the correctness of the notion of absurdity and as to the value of a proof for freedom from contradiction by means of this notion. But nevertheless it may be granted that this foundation is at least more satisfactory than the ordinary platonistic interpretation …'
   [Gödel, unpublished draft for *1933o][1]

8. 'All functions that deserve to be called effective must at least be definable in a way that is persistent with expansions of the universe of types'
   [Rathjen (2009), 'The constructive Hilbert program and the limits of Martin-Löf Type Theory', p.427]

1 All works of Kurt Gödel used with permission.

9. 'If mathematics consists of mental constructions, then every mathematical theorem is the expression of a result of a successful construction. The proof of the theorem consists in this construction itself, and the steps of the proof are the same as the steps of the mathematical construction.'
[Heyting (1958), 'Blick von der intuitionistische Warte', *Dialectica*, p.107]

10. 'Just as, in general, well-ordered species are produced by means of the two generating operations from primitive species (Brouwer 1926, p. 451), so, in particular, mathematical demonstrations [Beweisführungen] are produced by means of the two generating operations from null elements and elementary inferences [Elementarschlüssen] that are immediately given in intuition (albeit subject to the restriction that there always occurs a last elementary inference). These *mental* mathematical proofs [*gedanklichen* Beweisführungen] that in general contain infinitely many parts [Glieder] must not be confused with their linguistic accompaniments, which are finite and necessarily inadequate, hence do not belong to mathematics.

The preceding remark contains my main argument against the claims of Hilbert's metamathematics.'
[Brouwer (1927), 'Über Definitionsbereiche von Funktionen', *Mathematische Annalen* 97, p. 64, footnote 8, original emphasis]

11. Brouwer defines species as 'properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it, relations of equality having to be symmetric, reflexive and transitive; mathematical entities previously acquired for which the property holds are called the elements of the species.'
[Brouwer (1952), 'Historical background, principles and methods of intuitionism', *South-African Journal of Science* 49, p.142]

12. The existence of a species P amounts to the existence of a function that assigns to each existing object a the hypothesis that it has the property P, the hypothesis $P(a)$. The domain of that function is *every* existing object (all objects 'previously acquired') because for each of them the proposition that it has that property is meaningful (whether

that proposition turns out to be true, false, or undecided). The existence of this function is implicit in the concept of species.

13. 'With regard to this definition of species we have to remark firstly that, during the development of intuitionistic mathematics, some species will have to be considered as being re-defined time and again in the same way, secondly that a species can very well be an element of another species, but never an element of itself.'
[Brouwer (1952), 'Historical background, principles and methods of intuitionism', *South-African Journal of Science* 49, p.142]

14. '[A]ccording to current logical assumptions, there are what we may call self-reproductive processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all the terms having the said property into a whole'
[Russell (1906), 'On some difficulties in the theory of transfinite numbers and order types', *Proc. London Math. Society*, p.36]

15. '[H]ere we call a set denumerably unfinished if it has the following properties: we can never construct in a well-defined way more than a denumerable subset of it, but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted to the original set. But from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention.'
[Brouwer (1907), *Over de grondslagen der wiskunde*, p.148]

16. There exists a function f that transforms any element of the species of demonstration objects of A into an element of the species of demonstration objects of B.

17. There exists an infinite sequence of species of demonstration objects of A indexed by their stage of definition m and an infinite sequence of functions f_m, each extended by the next, such that f_m transforms any element of the species of demonstration objects of A at stage m into an

element of the species of demonstration objects of B at stage m.

$$
f : A \to B \quad = \quad
\begin{array}{ccccccccc}
A_1 & \subseteq & A_2 & \subseteq & A_3 & \cdots & A_m & \cdots \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_m & \\
B_1 & \subseteq & B_2 & \subseteq & B_3 & \cdots & B_m & \cdots
\end{array}
$$

18. A function is parametrically polymorphic if its arguments need not each be of one fixed type, but may come from a family of types, on which the function however acts uniformly; the types of the arguments are considered to be parameters, and with different instantiations of these parameters the function takes a different shape.

19. In the case of Brouwerian implication, as reconstructed here, a function f that proves an implication $A \to B$ is likewise parametrically polymorphic. The family of argument types it can act on consists of the species of demonstration objects of A at stage m, for all m; the function f has, so to speak, infinitely many types at once. Depending on A, the arguments of the function f may themselves be polymorphic functions (iteration of implication), but all polymorphic functions used here are bounded in that their domains are limited to demonstration objects that have been 'previously acquired'.

20. 'With this notation it is not possible to extend the domain of a mapping without changing the name of the mapping. If one has defined e.g. the function sin on the domain of the real numbers, and wishes to extend it to the domain of the complex numbers, one from now on needs to refer to that which so far was called sin by sin $\upharpoonright$ (real numbers)'
[Heyting (1930), 'Die formalen Regeln der intuitionistischen Logik (2)', p.66, trl. mine]

21. 'Definition 1. A species is a property which mathematical entities can be supposed to possess (L. E. J. Brouwer 1918, p. 4; 1924, p. 245; 1952, p. 142).

    Definition 2. After a species S has been defined, any mathematical entity which has been *or might have been* defined before S and which satisfies the condition S, is a member of the species S.'
[Heyting (1956), section 3.2.1, emphasis mine]