A lower semicontinuity result for a free discontinuity functional with a boundary term

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joint work with G. Dal Maso and R. Toader

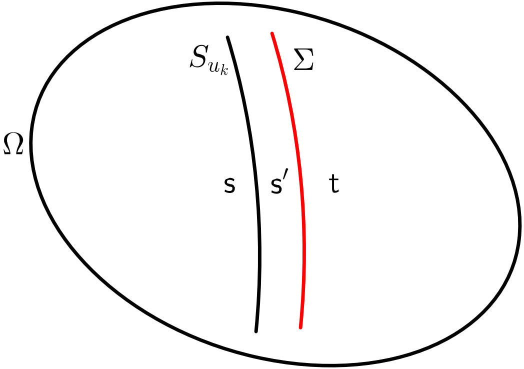
Problem setting

Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz buondary and let $p \in (1, +\infty)$. We study the lower semicontinuity in $GSBV^p(\Omega; \mathbb{R}^m)$ of the functional

$$\mathcal{F}(u) := \int_{S_u \setminus \Sigma} \psi(x, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g(x, u^+, u^-) \, \mathrm{d}\mathcal{H}^{n-1} \,,$$

where $\Sigma \subseteq \overline{\Omega}$ is an orientable Lipschitz manifold of dimension $n-1$, S_u denotes the jump set of u, ν_u stands for the approximate unit normal to S_u, and u^+ and u^- are the traces of u on the two sides of Σ, defined accordingly to the orientation of the unit normal ν_Σ to Σ.

Main result



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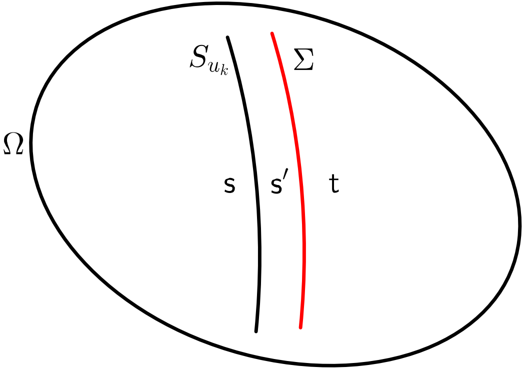

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along the sequence: $\approx g(x, s', t) + \psi(x, \nu_\Sigma(x))$,

at the limit: $g(x, s, t)$.

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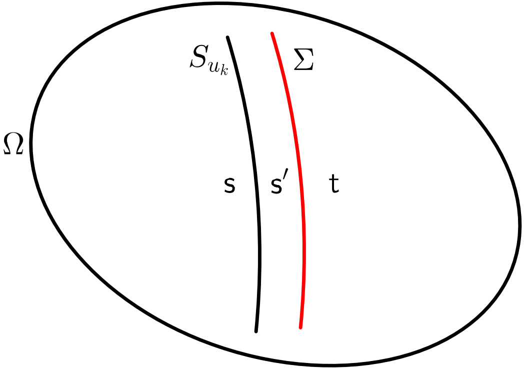

Let $x \in \Sigma$:

along the sequence: $\approx g(x, s', t) + \psi(x, \nu_\Sigma(x))$,

at the limit: $g(x, s, t)$.

$$\implies \quad g(x, s, t) \leq g(x, s', t) + \psi(x, \nu_\Sigma(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma \text{ and every } s, s', t \in \mathbb{R}^m.$$

Main result

Assume that $\psi\colon \overline{\Omega} \times \mathbb{R}^n \to [0, +\infty)$ is a continuous function such that:

- $\psi(x, \cdot)$ is a norm in $\mathbb{R}^n$ for every $x \in \overline{\Omega}$;
- there exist $0 < c_1 \leq c_2$ such that for every $x \in \overline{\Omega}$ and every $\nu \in \mathbb{R}^n$

$$c_1|\nu| \leq \psi(x, \nu) \leq c_2|\nu| \,.$$

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Suppose that $g\colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfies:

- g is Borel;
- $g(\cdot, 0, 0) \in L^1(\Sigma)$;
- $g(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R}^m \times \mathbb{R}^m$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$;
- for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, s', t, t' \in \mathbb{R}^m$:

$$g(x, s, t) \leq g(x, s', t) + \psi(x, \nu_\Sigma(x)) \quad \text{and} \quad g(x, s, t) \leq g(x, s, t') + \psi(x, \nu_\Sigma(x)) .$$

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Theorem (A., Dal Maso, Toader)

Under the above assumptions on ψ and g, the functional $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega; \mathbb{R}^m)$.

Relaxation result

Assume that $g\colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function such that:

- $g(x, \cdot, \cdot)$ is uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$;
- there exists $a \in L^1(\Sigma)$ such that $|g(x, s, t)| \leq a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$.

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Theorem (A., Dal Maso, Toader)

For every $u \in GSBV^p(\Omega;\mathbb{R}^m)$

$$sc^-\mathcal{F}(u) = \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1},$$

where we have set

$$g_{12}(x,s,t) := \{g_1(x,s,t)\,, \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x))\}\,,$$
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Remark

For $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, s', t, t' \in \mathbb{R}^m$

$$g_{12}(x,s,t) \leq g_{12}(x,s',t) + \psi(x,\nu_\Sigma(x)) \quad \textit{and} \quad g_{12}(x,s,t) \leq g_{12}(x,s,t') + \psi(x,\nu_\Sigma(x))\,.$$

Let $W\colon \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ and $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory functions such that

- $W(x, \cdot)$ is quasiconvex;
- there exist $0 < a_1 \leq a_2$ and $b_1, b_2 \in L^1(\Omega)$ such that

$$a_1|\xi|^p - b_1(x) \leq W(x, \xi) \leq a_2|\xi|^p + b_2(x)\,;$$

- there exist $q \in (1, +\infty)$, $0 < a_3 \leq a_4$, and $b_3, b_4 \in L^1(\Omega)$ such that

$$a_3|s|^p - b_3(x) \leq f(x, s) \leq a_4|s|^q + b_4(x)\,.$$

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$$a_3|s|^p - b_3(x) \le f(x,s) \le a_4|s|^q + b_4(x)\,.$$

We define

$$\mathcal{G}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$ and $\mathcal{G}(u) := +\infty$ if $u \in L^q(\Omega;\mathbb{R}^m) \setminus GSBV^p(\Omega;\mathbb{R}^m)$.

Theorem (A., Dal Maso, Toader)

$$sc^{-}\mathcal{G}(u) = \int_{\Omega} W(x, \nabla u)\,\mathrm{d}x + \int_{\Omega} f(x,u)\,\mathrm{d}x + \int_{S_u \setminus \Sigma} \psi(x, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g_{12}(x, u^{+}, u^{-})\,\mathrm{d}\mathcal{H}^{n-1}$$

for $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$ *and* $sc^{-}\mathcal{G}(u) = +\infty$ *if* $u \in L^q(\Omega;\mathbb{R}^m) \setminus GSBV^p(\Omega;\mathbb{R}^m)$.

Thanks for your attention!