Dirichlet-Neumann shape optimization problems

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In shape optimization problems we have in general a shape functional $F(\Omega)$ that has to be optimized among domains Ω of $\mathbf{R}^d$ belonging to some suitable admissible class $\mathcal{A}$.

Concerning the cost functional $F(\Omega)$ two types are interesting and should be considered:

- **integral functionals**;

- **spectral functionals**.

Integral functionals Given a fixed domain D and a right-hand side $f \in L^2(D)$ we consider the solution u_Ω of the elliptic PDE

$$-\Delta u = f(x) \text{ in } \Omega, \qquad u \in H_0^1(\Omega).$$

The integral cost functionals we consider are of the form

$$F(\Omega) = \int_D j\Big(x, u_\Omega(x), \nabla u_\Omega(x)\Big)\, dx$$

where j is a suitable integrand that we assume convex in the gradient variable and bounded from below as

$$j(x, s, z) \geq -a(x) - c|s|^2$$

with $a \in L^1(D)$ and c smaller than the first eigenvalue of $-\Delta$ on $H_0^1(D)$. In particular, the energy $\mathcal{E}_f(\Omega)$ defined by

$$\mathcal{E}_f(\Omega) = \inf_{u \in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - f(x) u \right) dx$$

belongs to this class since, integrating by parts its Euler-Lagrange equation, we have

$$\mathcal{E}_f(\Omega) = -\frac{1}{2} \int_\Omega f(x) u_\Omega \, dx$$

which corresponds to the integral functional above with

$$j(x, s, z) = -\frac{1}{2} f(x) s.$$

Spectral functionals For every admissible domain Ω we consider the spectrum $\lambda(\Omega)$ of the Dirichlet Laplacian $-\Delta$ on $H_0^1(\Omega)$.

If Ω has a finite measure $-\Delta$ has a compact resolvent and so its spectrum $\lambda(\Omega)$ is discrete:

$$\lambda(\Omega) = \Big(\lambda_1(\Omega), \lambda_2(\Omega), \dots\Big),$$

where $\lambda_k(\Omega)$ are the eigenvalues counted with their multiplicity.

The spectral cost functionals we consider are of the form

$$F(\Omega) = \Phi\Big(\lambda(\Omega)\Big)$$

where $\Phi : \mathbf{R}^{\mathbf{N}} \to \overline{\mathbf{R}}$ is a given function. For instance, taking $\Phi(\lambda) = \lambda_k$ we obtain

$$F(\Omega) = \lambda_k(\Omega).$$

We say that Φ is continuous (resp. lsc) if

$$\lambda_k^n \to \lambda_k \ \forall k \Longrightarrow \Phi(\lambda_n) \to \Phi(\lambda)$$
$$\Big(\text{resp. } \Phi(\lambda) \leq \liminf_n \Phi(\lambda_n)\Big).$$

Concerning the admissible class $\mathcal{A}$ of competing domains, we consider the one which consists of all subdomains of a given domain D of $\mathbf{R}^d$, with a constraint on its Lebesgue measure $|\Omega|$, without other extra geometric constraints.

It is known that the addition of extra conditions on the competing domains (as convexity, equi-Lipschitz condition, . . .) introduces an additional compactness that leads to the existence of an optimal solution.

The most studied case in the literature is the one, in both the types of $F(\Omega)$ above, where the Laplace operator $-\Delta$ is considered with Dirichlet conditions on the free boundary $\partial\Omega$.

In this case the most general available result is when the competing domains are a priori supposed inside a given bounded region D.

Here the word "domain" has to be intended as "quasi-open set". An interesting (and difficult) issue is to investigate about the regularity of the optimal solutions.

Theorem (Buttazzo-Dal Maso, ARMA '93): assume that

• *F is monotone decreasing for the set inclusion;*

• *F is γ-lower semicontinuous.*

Then the shape optimization problem

$$\min\Big\{F(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

admits a solution.

In general the solution is a quasi-open set and very little is known about its regularity in the case of particular shape functionals F.

It has to be mentioned that, in the case of spectral functionals, several recent results are now available (Bucur, Mazzoleni, Pratelli, Velichkov, . . .):

• allowing the region D to be the whole $\mathbf{R}^d$;

• showing that the optimal sets are bounded and with finite perimeter;

• showing that in particular cases of functionals F the optimal domains are actually open sets.

The monotonicity assumption in the existence theorem above is important, without it in general the solution has to be found in the relaxed class of capacitary measures, that is Borel measures μ with values in $[0, +\infty]$ and such that

$$\mu(E) = 0 \text{ whenever } \operatorname{cap}(E) = 0.$$

On the other hand, the γ-l.s.c. assumption is very light, since the γ-convergence is rather strong and most of the interesting shape functionals are actually γ-continuous.

We consider now shape optimization problems where the boundary of the unknown set Ω has a part where the Dirichlet condition is imposed and the remaining part is subjected to the Neumann condition. Two cases are considered:

• The Neumann condition acts on a fixed part and the Dirichlet condition acts on the free boundary;

• The Dirichlet condition acts on a fixed part and the Neumann condition acts on the free boundary.

Fixed Neumann, free Dirichlet

This case is quite similar to the previous one where the Dirichlet condition was imposed on all the boundary $\partial\Omega$. The precise formulation of the problem requires a given bounded set $D \subset \mathbf{R}^d$ and the Sobolev spaces

$$H_0^1(\Omega; D) = \Big\{u \in H^1(D) \ : \ u = 0 \text{ q.e. on } D \backslash \Omega\Big\}.$$

The minimization problem

$$\min\left\{ \int_D \left(\frac{1}{2}|\nabla u|^2 - f(x)u\right) dx \ : \ u \in H_0^1(\Omega; D)\right\}$$

then corresponds to the PDE

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \cap D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \cap \partial D. \end{cases}$$

Similarly, the first eigenvalue is

$$\lambda_1(\Omega; D) = \min_{u \in H_0^1(\Omega; D)} \left\{ \int_D |\nabla u|^2 \, dx \ : \ \int_D u^2 \, dx = 1 \right\}$$

and the spectrum $\sigma(\Omega; D)$ is defined accordingly. The γ_D-convergence is also defined in a similar way. The result in this case is similar to the Buttazzo-Dal Maso one:

Theorem. Assume that

• F *is monotone decreasing for the set inclusion;*

• F *is* γ_D*-lower semicontinuous.*

Then the shape optimization problem

$$\min\Big\{F(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

admits a solution.

In [B.-Velichkov] we studied this problem and we called it spectral drop problem, since it reminds the classical drop problems, where the total variation functional replaces the shape functional F.

Take now $F(\Omega) = \lambda_1(\Omega; D)$. The following properties for an optimal domain Ω hold.

• Ω must touch the boundary of D; precisely, $\mathcal{H}^{d-1}(\partial\Omega \cap \partial D) > 0$.

• $\partial\Omega$ intersects ∂D orthogonally.

• Similar if D is the complement of a bounded set and if D is an unbounded convex domain.

• Nonexistence for general unbounded domains D, for instance if D is the complement of an unbounded strictly convex set.

• Another variant consists in adding the term

$$\frac{k}{2}\int_{\partial D} u^2\,d\mathcal{H}^{d-1}$$

in the energy, similarly to what is made in the classical drop problem; the constant k has to be not too negative, precisely

$$k > -\inf_{\substack{u\in H_0^1(\Omega;D)\\ |\Omega|\le 1}} \Big\{\int_D |\nabla u|^2\,dx \;:\; \|u\|_{L^2(\partial D)} = 1\Big\}.$$

In this way the Neumann condition becomes the Robin one, and the properties above still hold, except of course the one of orthogonal intersection with ∂D.

Fixed Dirichlet, free Neumann

This is a quite different type of problem, presenting several additional difficulties, mainly due to the fact that, even for the simple situation of a shape functional like the first eigenvalue, the monotonicity assumption does not hold. Let us consider as before a given domain D and a fixed Dirichlet region K; the corresponding Sobolev space is

$$H^1(D;K) = \Big\{u \in H^1(D) \ : \ u = 0 \text{ q.e. on } K\Big\}$$

and the first eigenvalue is

$$\mu_1(\Omega; K) = \inf_{u \in H^1(D;K)} \left\{ \int_\Omega |\nabla u|^2 dx \ : \ \int_\Omega |u|^2 dx = 1 \right\}.$$

Note that if Ω is smooth the related PDE is

$$\begin{cases} -\Delta u = \mu_1 u \text{ in } \Omega \\ u = 0 \text{ on } K, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

From now on let us take, for simplicity, the case of the shape functional $F(\Omega) = -\mu_1(\Omega; K)$, so that the question is to maximize $\mu_1(\Omega; K)$ under a volume constraint. In fact, minimizing $\mu_1(\Omega; K)$ is trivial, since a disconnected domain Ω gives zero as its first eigenvalue.

We consider then the problem

$$\max\Big\{\mu_1(\Omega;K)\ :\ \Omega\subset D,\ |\Omega|\ge m\Big\}$$

or, more conveniently, its analogous

$$\max\Big\{|\Omega|\ :\ \Omega\subset D,\ \mu_1(\Omega;K)\ge\mu\Big\}$$

where μ is a given positive number. Since

$$\inf\left\{\int_\Omega\Big(|\nabla u|^2-\mu|u|^2\Big)\,dx\ :\ u\in H^1(D;K)\right\}$$
$$=\begin{cases}0 & \text{if } \mu_1(\Omega;K)\ge\mu\\ -\infty & \text{if } \mu_1(\Omega;K)<\mu\end{cases}$$

the shape optimization problem above can be written as

$$\max_{\Omega\subset D}\ \min_{u\in H^1(D,K)}\left(|\Omega| + \int_\Omega \left(|\nabla u|^2 - \mu|u|^2\right)dx\right).$$

Since

$$\max_{\Omega\subset D}\ \min_{u\in H^1(D,K)} \le \min_{u\in H^1(D,K)}\ \max_{\Omega\subset D}$$

a larger quantity is

$$\min_{u\in H^1(D,K)}\ \max_{\Omega\subset D}\left(|\Omega| + \int_\Omega \left(|\nabla u|^2 - \mu|u|^2\right)dx\right).$$

Enlarging the class of admissible choices from the domains Ω to the relaxed class of density functions $0 \le \theta(x) \le 1$ a still larger quantity is

$$\min_{u\in H^1(D,K)} \max_{0\le\theta\le 1} \left(\int_D \theta\, dx + \int_D \theta \Big(|\nabla u|^2 - \mu |u|^2 \Big)\, dx \right).$$

The max in θ is easy to compute and gives

$$\int_D \Big(|\nabla u|^2 - \mu |u|^2 + 1 \Big)^+ dx$$

so that the problem is reduced to

$$\min_{u\in H^1(D,K)} \int_D \Big(|\nabla u|^2 - \mu |u|^2 + 1 \Big)^+ dx.$$

If $\bar{u}$ is a solution to this last problem a candidate optimal domain $\bar{\Omega}$ is the set

$$\bar{\Omega} = \Big\{ |\nabla \bar{u}|^2 - \mu |\bar{u}|^2 + 1 > 0 \Big\}.$$

Theorem. *The auxiliary problem*

$$\min_{u\in H^1(D,K)} \int_D \left(|\nabla u|^2 - \mu|u|^2 + 1\right)^+ dx.$$

is equivalent to a double obstacle problem

$$\min_{0\le w\le 1} \left\{ \int_D \left(\frac{1}{\mu}|\nabla w|^2 - w^2 + 1\right) dx \ :\ w \in H^1(D;K)\right\}.$$

whose solution $\bar{w}$ *identifies the optimal domain through* $\bar{\Omega} = \{\bar{w} < 1\}$*.*

The properties of the solution are:

• If K is the ball of radius r_0, then $\bar{\Omega}$ is the ball of radius $R(r_0)$ for a suitable $R(r_0)$ explicitly computable.

• If K is bounded, then $\bar{\Omega}$ is bounded.

• The solution $\bar{w}$ is continuous and so the optimal set $\bar{\Omega}$ is open.

• The regularity of $\bar{\Omega}$ is the same than for double obstacle problem and depends on K.

Similar results for the torsion shape functional

$$\mathcal{E}(\Omega) = \inf \left\{ \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - u \right) dx \right\}$$

and (hopefully...) for the energy shape functional

$$\mathcal{E}_f(\Omega) = \inf \left\{ \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - f(x) u \right) dx \right\}.$$

Work(s) in progress with D. Bucur and B. Velichkov.

Happy birthday

Luisa