Heteroclinics for a mean curvature problem in Lorentz-Minkowski space

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A quasilinear bistable equation in cylinders

Let $\omega \subset \mathbb{R}^{N-1}$ be a bounded domain $(N \geq 1)$. We consider the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = W'(u), \quad \text{in } \mathbb{R} \times \omega,$$

with the boundary conditions

$$\lim_{x \to \pm\infty} \big(u(x,y), \partial_x u(x,y)\big) = (\pm 1, 0), \quad \text{uniformly in } y \in \omega,$$

where W is a *double well potential*

- $W \in C^1(\mathbb{R})$,
- $W(-1) = W(1) = 0$ and $W(s) > 0$ if $s \neq \pm 1$.

Example: the potential for the Allen-Cahn equation $W(s) = \frac{1}{4}\left(1 - s^2\right)^2$

Goal: We look for solutions connecting the equilibria $u = -1$ and $u = +1$ along the first coordinate.

Minkowski curvature operator

Newton's Second Law of Motion:

$$F = ma = \frac{d}{dt}(mv) = \big(mu'\big)'$$

Special Theory of Relativity: the mass of a body increases with velocity

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0 =$ rest mass and $c = 3 \times 10^8 m/s =$ the speed of light.
With the normalization $m_0 = c = 1$, we recover the equation

$$F = \left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)'$$

[The Feynman Lectures on Physics 1964]

Riemannian Geometry: Represents the local mean curvature of hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{N+1}$ with coordinates $(x_1, \dots, x_N, t)$ and the metric $\sum_{j=1}^{N}(dx_j)^2 - (dt)^2$.

[Bartnik Simon 1982]

A minimal energy solution

Our approach will be variational:

We will look for a transition from the equilibrium state $u = -1$ to the equilibrium state $u = +1$ minimising the *energy functional of Ginzburg-Landau type*

$$J(u) = \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u|^2}\, d\bar{x} + \int_{\mathbb{R}\times\omega} W(u)\, d\bar{x}$$

in the functional space

$$\mathcal{X} = \Big\{ u \in W^{1,\infty}(\mathbb{R}\times\omega) \mid \|\nabla u\|_\infty \leq 1 \ \text{ and } \lim_{x\to\pm\infty} u(x,y) = \pm 1 \text{ uniformly in y} \in \omega \Big\}.$$

Our main result

Theorem

The energy functional J attains its infimum in $\mathcal{X}$. Its minimiser u depends only on the first variable $x \in \mathbb{R}$, is non-decreasing and is the unique solution, up to translations, of the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = W'(u), \qquad \text{in } \mathbb{R}\times\omega,$$

satisfying the boundary conditions

$$\lim_{x\to\pm\infty}\big(u(x,y),\partial_x u(x,y)\big) = (\pm 1, 0), \qquad \text{uniformly in } y\in\omega.$$

Moreover, u satisfies the conservation of energy law

$$1 - \frac{1}{\sqrt{1-|u'|^2}} + W(u) = 0.$$

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satisfying the boundary conditions

$$\lim_{x\to\pm\infty} \big(u(x), u'(x)\big) = (\pm 1, 0).$$

Moreover, u satisfies the conservation of energy law

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Idea of the proof

The proof will be done in 3 steps:

STEP 1 we use a monotone rearrangement to prove that minimising sequences of J in $\mathcal{X}$ can be assumed to depend only on the first variable,

STEP 2 we prove the existence of a one-dimensional minimiser with $\|\nabla u\|_\infty < 1$ and a law of conservation of energy,

STEP 3 we use a regularisation scheme to exclude the existence of N-dimensional minimisers.

Uniqueness (up to translations) follows from the conservation of energy.

STEP 1: Monotone rearrangement

Idea: to reorganize the level sets of u in order to obtain a one-dimensional, non-decreasing function $u^\star$ passing through zero.

For any function $u\colon \mathbb{R}\times\omega \to [-1,1]$, we define the *level sets* of u by

$$\Omega_c = \begin{cases} \{(x,y)\in\mathbb{R}\times\omega \mid c < u(x,y) < 0\} & \text{if } -1 \le c < 0, \\ \{(x,y)\in\mathbb{R}\times\omega \mid 0 \le u(x,y) \le c\} & \text{if } 0 \le c \le 1. \end{cases}$$

We define a *rearrangement* $u^\star\colon \mathbb{R}\times\omega\to[-1,1]$ of u through its level sets

$$\Omega_c^\star = \begin{cases} \left]-\frac{m_N(\Omega_c)}{m_{N-1}(\omega)}, 0\right[\times\omega & \text{if } -1\le c<0, \\ \left[0, \frac{m_N(\Omega_c)}{m_{N-1}(\omega)}\right]\times\omega & \text{if } 0\le c\le 1. \end{cases}$$

This generalisation of the classical monotone rearrangement for functions of a single variable was introduced and discussed in

[Carbou 1995, Farina 1999, Alberti 2000]

STEP 1: Monotone rearrangement

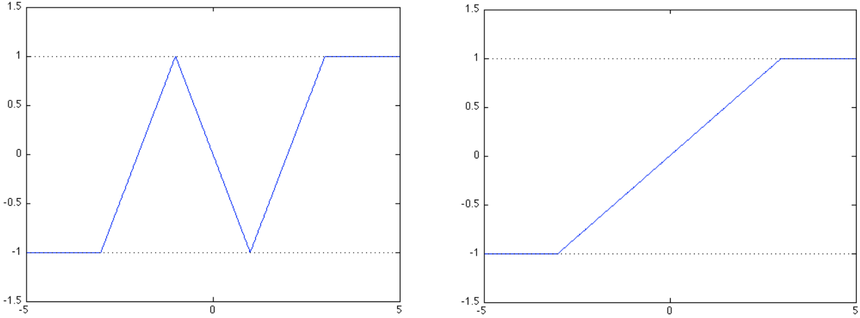

Figure: On the left a function $u \in \mathcal{X}$ and its rearrangement $u^\star$ on the right.

- The level sets of $u^\star$ are all cylinders and $m_N(\Omega_c^\star) = m_N(\Omega_c)$.
- $u^\star$ is one-dimensional, non-decreasing, continuous and $u^\star(0, y) = 0$ for all $y \in \omega$. Thus, we will use the notation $u^\star(x) = u^\star(x, y)$.

Step 1: Properties of the rearrangement

Theorem (Pólya-Szegö type theorem [Alberti 2000, Theorem 2.10])

Let $g\colon [0,+\infty) \to [0,+\infty)$ be convex, $g(0)=0$ and strictly increasing. Then for every $u \in \mathcal{X}$ taking values in $[-1,1]$, we have

$$\int_{\mathbb{R}\times\omega} g(|\nabla u^\star|) \leq \int_{\mathbb{R}\times\omega} g(|\nabla u|).$$

Moreover, if the left-hand side is finite, equality holds if and only if there exists $a \in \mathbb{R}$ such that $u(x+a) = u^\star(x)$ for every $x \in \mathbb{R}$.

Lemma (Bound on the gradient)

Assume that $u \in \mathcal{X}$, takes values in $[-1,1]$ and $|\nabla u| \in L^2(\mathbb{R}\times\omega)$. Then $u^\star \in \mathcal{X}$ and

$$\|\nabla u^\star\|_\infty \leq \|\nabla u\|_\infty .$$

Step 1: Properties of the rearrangement

Theorem (Kinetic energy)

For all $u \in \mathcal{X}$ taking values in $[-1,1]$ and such that $|\nabla u| \in L^2(\mathbb{R} \times \omega)$, we have

$$\int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u^\star|^2}\, d\bar{x} \;\; \leq \;\; \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u|^2}\, d\bar{x}.$$

Moreover, if the left-hand side is finite and $\|\nabla u\|_\infty < 1$, we have equality if and only if there exists $a \in \mathbb{R}$ such that $u(x+a) = u^\star(x)$ for all $x \in \mathbb{R}$.

Step 1: Idea of the proof

[Alberti 2000, Theorem 2.10] can be applied to the truncated functions g_n

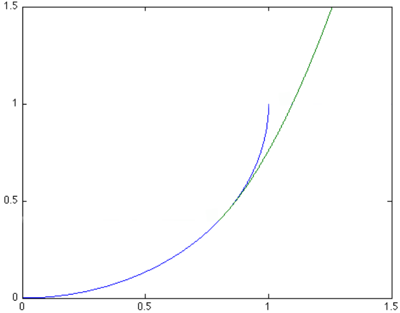

Figure: blue = the potential $1 - \sqrt{1 - s^2}$, green = the modified $g_n(s)$.

For all $n \in \mathbb{N}$, $\displaystyle\int_{\mathbb{R}\times\omega} g_n(|\nabla u^\star|)\, d\bar{x} \;\le\; \int_{\mathbb{R}\times\omega} g_n(|\nabla u|)\, d\bar{x}$

$|\nabla u| \in L^2$ + Lebesgue's dominated convergence $\Longrightarrow$

$$\int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u^\star|^2}\, d\bar{x} \;\le\; \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u|^2}\, d\bar{x}.$$

□

STEP 1: Properties of the rearrangement

Theorem (Cavalieri's Principle)

For every continuous function $F\colon [-1,1] \to \mathbb{R}$ *and every function* $u\colon \overline{\Omega} \to [-1,1]$, *we have*

$$\int_{\overline{\Omega}} F(u)\, d\bar{x} = \int_{\overline{\Omega}^\star} F(u^\star)\, d\bar{x}.$$

Step 1: Properties of the rearrangement

Combining the previous results, we obtain the following theorem:

Theorem (Energy of the rearrangement)

For all $u \in \mathcal{X}$ taking values in $[-1,1]$ such that $|\nabla u| \in L^2(\mathbb{R} \times \omega)$, there exists $u^\star \in \mathcal{X}$ depending only on x and non-decreasing such that

$$J(u^\star) = \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u^\star|^2}\, d\bar{x} + \int_{\mathbb{R}\times\omega} W(u^\star)\, d\bar{x} \leq \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u|^2}\, d\bar{x} + \int_{\mathbb{R}\times\omega} W(u)\, d\bar{x} = J(u).$$

Moreover, if the left-hand side is finite and $\|\nabla u\|_\infty < 1$, we have equality if and only if there exists $a \in \mathbb{R}$ such that $u(x+a) = u^\star(x)$ for all $x \in \mathbb{R}$.

Step 2: Existence of a one-dimensional minimiser

Let $(u_k) \subset \mathcal{X}$ be a minimising sequence such that $-1 \leq u_k \leq 1$.

Passing to the rearranged functions $u_k^\star$, if necessary, we may assume that the functions u_k are one-dimensional, non-decreasing and that $u_k(0) = 0$.

For this sequence $J(u_k) = m_{N-1}(\omega) J_1(u_k)$, where

$$J_1(u) = \int_{\mathbb{R}} 1 - \sqrt{1 - |u'(x)|^2}\, dx + \int_{\mathbb{R}} W(u(x))\, dx.$$

Claim 1: J_1 attains its infimum in the corresponding one-dimensional space

$$\mathcal{X}_1 = \left\{ u \in W^{1,\infty}(\mathbb{R}) \mid \left\| u' \right\|_\infty \leq 1 \text{ and } \lim_{x \to \pm\infty} u(x) = \pm 1 \right\}.$$

Step 2: Existence of a one-dimensional minimiser

We have: $\sup \|u_k\|_\infty \le 1$, $\sup \|u_k'\|_\infty \le 1$ and $\sup \|u_k'\|_{L^2}$ is bounded.

Ascoli-Arzelà's Theorem and a diagonal procedure $\implies \exists\, u \in W^{1,\infty}(\mathbb{R})$ such that

$$u_k \to u \text{ in } C_{loc}(\mathbb{R}) \text{ and } u_k' \rightharpoonup u' \text{ in } L^2(\mathbb{R}).$$

u is non-decreasing, bounded in $[-1,1]$, $u(0) = 0$ and $\|u'\|_\infty \le 1$.

By weak lower semicontinuity of the kinetic part and Fatou's lemma

$$\begin{aligned} J_1(u) &\le \liminf \int_{\mathbb{R}} 1 - \sqrt{1 - |u_k'|^2}\, dt + \liminf \int_{\mathbb{R}} W(u_k)\, dt \\ &= \lim J_1(u_k) = \inf_{\mathcal{X}_1} J_1. \end{aligned}$$

In addition, $\lim_{x \to \pm\infty} u(x) = \pm 1 \implies u \in \mathcal{X}_1$. Therefore, $J_1(u) = \min_{\mathcal{X}_1} J_1$.

STEP 2: Existence of a one-dimensional minimiser

Claim 2: If u is a minimiser of J_1 in $\mathcal{X}_1 \implies \|u'\|_\infty < 1$

Proof of the claim

For fixed $x_0 < x_1$ and $0 < \theta < 1$, we define the stretching u_θ of u as

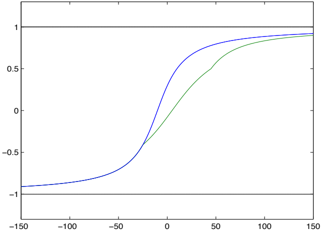

Figure: Graphs of u (in blue) and u_θ (in green).

It is easily seen that $u_\theta \in \mathcal{X}_1$ and $J_1(u) \leq J_1(u_\theta)$.

Step 2: Existence of a one-dimensional minimiser

From $J_1(u) \leq J_1(u_\theta)$ we obtain an uniform estimate on the derivative of u

$$|u'| \leq \sqrt{1 - \frac{1}{\left(1 + W(u)\right)^2}} \leq \sqrt{1 - \frac{1}{\left(1 + \max\limits_{[-1,1]} W\right)^2}} < 1.$$

$$\text{So, } \|u'\|_\infty \leq 1 - \varepsilon < 1.$$

Now, we can use the weak formulation $\implies u \in C^2(\mathbb{R})$, is a solution of the BVP and satisfies the conservation of energy law

$$1 - \frac{1}{\sqrt{1 - |u'|^2}} + W(u) = 0.$$

Uniqueness (up to translations) follows from the law of conservation of energy.

STEP 3: Excluding N-dimensional minimisers

Claim 3: The functional J has no minimisers with $\|\nabla u\|_\infty = 1$.

Proof of the claim

Suppose that $u_0 \in \mathcal{X}$ is a minimiser of J with $\|\nabla u\|_\infty = 1$.

Recall the truncated function g_n

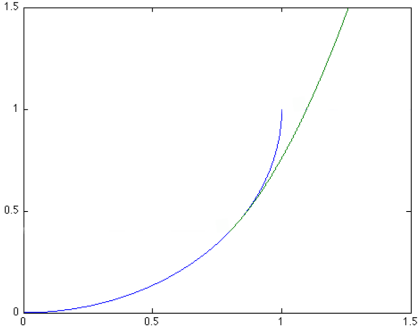

Figure: blue = the potential $1 - \sqrt{1 - s^2}$, green = the modified $g_n(s)$.

Step 3: Excluding N-dimensional minimisers

The Direct Method of the Calculus of Variations can be applied to the modified functional

$$J_n(u) = \int_{\mathbb{R}\times\omega} g_n(u')\ d\bar{x} + \int_{\mathbb{R}\times\omega} W(u)\ d\bar{x}.$$

Arguing as above, we see that

J_n has a one-dimensional minimiser $u_n \in \mathcal{X}$ with $\|u'\|_\infty < 1$.

For a good choice of n, the minimiser u_n is on the region where the functional was not modified. Therefore,

$$J(u_n) = J_n(u_n) \leq J_n(u_0) < J(u_0),$$

which contradicts the minimality of u_0. □

Conclusion

The functional

$$J(u) = \int_{\mathbb{R}\times\omega} 1 - \sqrt{1 - |\nabla u|^2} + W(u)\ d\bar{x}$$

has a unique (up to translations) one-dimensional minimiser in $\mathcal{X}$, which satisfies the estimate $\|u'\|_\infty \leq c < 1$ and is a classical solution of the BVP

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}} \right)' = W'(u), \quad \text{in } \mathbb{R},$$

$$\lim_{x \to \pm\infty} \big(u(x), u'(x)\big) = (\pm 1, 0).$$

Nonautonomous problems

The same ideas can be applied to problems of the form

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = a(t)W'(u), \qquad \text{in } \mathbb{R},$$

$$\lim_{x\to\pm\infty} \big(u(x), u'(x)\big) = (\pm 1, 0).$$

Theorem

- *IF* $0 \le a_1 \le a(t) \le a_2$ *and* $a(t) < a_2$ *in some nonempty set,*
- *OR IF* $a(t) \ge 0$ *is* T*-periodic, for some* $T > 0$,

then the energy functional

$$\mathcal{L}(u) = \int_{\mathbb{R}} 1 - \sqrt{1-|u'|^2} + a(t)\,W(u)\,dt$$

attains its infimum in the space

$$\mathcal{X}_1 = \Big\{u \in W^{1,\infty}(\mathbb{R}) \mid \big\|u'\big\|_\infty \le 1 \;\; \text{and} \;\; \lim_{x\to\pm\infty} u(x) = \pm 1\Big\}.$$

Nonautonomous problems

Lemma

If u is a minimiser of $\mathcal{L}$ in $\mathcal{X}_1$ and there exists $c > 0$ such that $a'(t) \leq ca(t)$ then $\|u'\|_\infty < 1$ and u is a solution of the nonautonomous BVP.

Idea of the proof

From $\mathcal{L}(u) \leq \mathcal{L}(u_\theta)$, we obtain

$$|u'(t)| \leq \sqrt{1 - \frac{1}{1 + a(t)W(u(t)) + \int_t^{+\infty} a'(s)W(u(s))\,ds}} \quad \text{a.e. } t \in \mathbb{R}.$$

u is a minimiser of $\mathcal{L}$ $\implies \displaystyle\int_{\mathbb{R}} a(t)W(u(t))\,dt \leq C$

$$\exists\, c > 0\colon a'(t) \leq ca(t) \implies$$

$$\int_t^{+\infty} a'(s)W(u)\,ds \leq \int_t^{+\infty} c\,a(s)W(u)\,ds < +\infty$$

So $\|u'\|_\infty \leq 1 - \varepsilon < 1$ and u is a solution of the nonautonomous BVP. □

A problem with symmetry

Theorem (Antisymmetric problem)

If

- $W(s)$ *and* $a(t)$ *are even functions,*
- $a(t)$ *is non-decreasing in* $[0, +\infty[$ *and* $\liminf_{t\to\infty} a(t) > 0$,
- *there exists* $c > 0$ *such that* $a'(t) \leq ca(t)$,

then there exists an antisymmetric heteroclinic to the nonautonomous BVP.

Idea of the proof

Minimisation in the subset of $\mathcal{X}_1$ of antisymmetric functions.

Open problem: Is the global minimiser of $\mathcal{L}$ in $\mathcal{X}_1$ antisymmetric ?

THANK YOU FOR YOUR ATTENTION !

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