Detection of a moving rigid body in a perfect fluid

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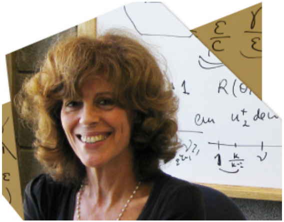
Celebrating the 65th anniversary of

Luísa MASCARENHAS



- To recover geometrical information (position and shape) about an a-priori unknown, static or moving body $S = S(t)$ immersed in an incompressible fluid.

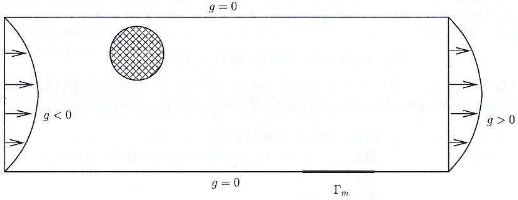


FIGURE 1. Moving obstacle in a pipeline.

- $g = \mathbf{v} \cdot \mathbf{n}$ denotes the input flow through the boundary Ω.
- To this end, we perform measurements (on velocity and stress forces) along the boundary of Ω.

Problem statement (II)

for both static and moving cases

- To simplify matters, we will start assuming the liquid to be a perfect fluid; its dynamics is hence governed by general incompressible Euler equations.
- Non-homogeneous Dirichlet boundary condition on $\partial\Omega$.
- Coupling conditions between Euler's equations and Newton's laws at the interfase : continuity of the normal velocity, conservation laws for linear and angular momentum.
- Let S be fixed. For g given, Cauchy forces are measured on a piece Γ_m of the boundary. The measurements are encoded in a certain boundary map

$$\Lambda_S : g \to \text{ Cauchy forces on } \; \Gamma_m$$

Desirable issues - Main questions
static case

- Identifiability result,

$$S_0 \neq S_1 \Longrightarrow \Lambda_{S_0} \neq \Lambda_{S_1}$$

- Stability result, that is, if two measures are close each other, then the rigid bodies are also close?.
- Algorithm and numerical reconstruction allow us to recover the volume and position of the unknown rigid body.

Let Ω be a smooth bounded set in $\mathbb{R}^N$ $(N=2)$ and let $S = S(t) \subset\subset \Omega$ be an unknown rigid body immersed in the liquid.

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= 0, \quad \mathrm{div}\ \mathbf{v} = 0 \quad \text{in} \quad \Omega \setminus \overline{S(t)} \\
\mathbf{v}\cdot\mathbf{n} &= (\mathbf{h}' + \omega(\boldsymbol{x} - \mathbf{h})^{\perp})\cdot\mathbf{n} \quad \text{on} \quad \partial S(t) \\
\mathbf{v}\cdot\mathbf{n} &= g \quad \text{on} \quad \partial\Omega \\
m\mathbf{h}''(t) &= \int_{\partial S(t)} p\,\mathbf{n}\,ds \\
J\omega'(t) &= \int_{\partial S(t)} (\boldsymbol{x} - \mathbf{h}(t))^{\perp}\cdot p\,\mathbf{n}\,ds
\end{aligned}$$

and for $t \geq 0$, where

- $\mathbf{h}(t)$: center of mass of $S(t)$
- ω : angular speed
- $\mathbf{n}$: outward unit normal vector, $\boldsymbol{x}^{\perp} = (-x_2, x_1)$ if $\boldsymbol{x} = (x_1, x_2)$

Outline

This lecture is organized as follows :

- Setting of the problem
- Identifiability results for a moving unknown obstacle in a perfect fluid
- Identifiability results for a moving unknown obstacle in a viscous fluid

Ghost solutions

A. Majda, *Comm. Pure Appl. Math.* **37**, 1984

- Euler equations do not exhibit any unique continuation property because of the existence of *ghost solutions* with compact support.
- A simple example is provided by the stationary solution $\mathbf{v}(x) = (\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$, where the stream function ψ is given by

$$\psi(x) = -\int_1^{|x|} \frac{1}{r}\left(\int_1^r s\, w(s)\, ds\right) dr$$

and the vorticity $w \in C^\infty(\mathbb{R}^+)$ is chosen so that $w(s) = 0$ for $s \geq 1$ and $\int_r^1 s w(s) ds = 0$ for $r \in (0, r_0)$, where $r_0 \in (0, 1)$.

- $\mathbf{v}(x)$ is supported in the set $\{r_0 \leq |x| \leq 1\}$. Thus, identifiability properties fail for Eulerian flows.

- Let us fix $t = t_0$ and focus on the determination of the position and velocity of $S(t_0)$ from one boundary measurement of the velocity at time t_0.
- We will hence ignore Newton's laws for $S(t)$ in our analysis.

- (**Moreover**) we will restrict ourselves to potential flows, i.e. flows for which

$$\mathbf{v} = \nabla \varphi$$

- and to spherical obstacles, say

$$S(t) = B_1(\mathbf{h}(t)) = \text{ ball of radius 1 and centered at } \mathbf{h}(t)$$

Potential flow model

Plugging $\mathbf{v} = \nabla\varphi$ in Euler's system results in a Laplace type-like system

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in} \quad \Omega \setminus S(t_0) \\ \frac{\partial\varphi}{\partial n} &= (\mathbf{h}' + \omega(\boldsymbol{x} - \mathbf{h})^{\perp}) \cdot \mathbf{n} \quad \text{on} \quad \partial S(t_0) \\ \frac{\partial\varphi}{\partial n} &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

Clearly,
measuring the normal component of the velocity $\mathbf{v}$ on one part of the boundary amounts to measuring the function φ itself.

Stop : "On va faire le point"

- Case $(\mathbf{h}', \omega) = (\mathbf{0}, 0)$ (static case)
  The boundary condition on $\partial S(t)$ simplifies to $\frac{\partial \varphi}{\partial n} = 0$, so the detection of S reduces to a quite classical problem :
  - Alessandrini, Díaz-Valenzuela 1996, *Siam J. Control Optim.*,
  - Alessandrini, Beretta, Rosset, Vessella 1999, *An. Sc. Sp. Pisa*
- What is new here? (Novelties) Essentially, the fact that the obstacle is moving, i.e., $(\mathbf{h}', \omega) \neq (\mathbf{0}, 0)$
- What makes the difference? The obstacle S may occupy different positions and undergo different velocities for a given Neumann data g.
- What to do? Further simplify our goal. We will just address the identifiability issue when the shape of the obstacle is known a priori.

Assume that $S(t) = B_1(\mathbf{h}(t))$. Then

$$\boldsymbol{x} - \mathbf{h} = -\mathbf{n} \quad \forall x \in \partial B_1(\mathbf{h}(t)),$$

and hence $(\boldsymbol{x} - \mathbf{h})^{\perp} \cdot \mathbf{n} = 0$. Setting $\boldsymbol{\ell} = \mathbf{h}'$, the system reads

$$\begin{cases} \Delta\varphi = 0 & \text{in} \quad \Omega \setminus \overline{B_1(\mathbf{h}(t))} \\ \dfrac{\partial\varphi}{\partial n} = \boldsymbol{\ell} \cdot \mathbf{n} & \text{on} \quad \partial B_1(\mathbf{h}(t)) \\ \dfrac{\partial\varphi}{\partial n} = g & \text{on} \quad \partial\Omega \end{cases}$$

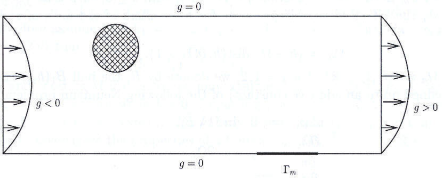


FIGURE 1. Moving obstacle in a pipeline.

Natural input flows g

Standard Neumann compatibility condition

$$\int_{\partial\Omega} g ds = 0$$

Linear input flows are excluded

Clearly, $g(x) = \boldsymbol{\ell} \cdot \mathbf{n}(x)$, with $\boldsymbol{\ell} \in \mathbb{R}^2$ a given vector, has to be excluded, for it may lead to the situation where the ball, which is surrounded by a fluid flowing at the same velocity ($\varphi(x, t) = \boldsymbol{\ell} \cdot x$), is not identifiable.

Theorem of Identifiability

Theorem 1

- *Define* $V = \mathrm{span}\,\{\mathbf{e}_1 \cdot \mathbf{n}, \mathbf{e}_2 \cdot \mathbf{n}\} \subset L^\infty(\partial\Omega)$.
- *Assume that* $g \in H^s(\partial\Omega) \setminus V$ *with* $s > 1/2$. *For* $i = 1, 2$, *choose* $(\boldsymbol{h}_i, \boldsymbol{\ell}_i) \in \Omega_{ad} \times \mathbb{R}^2$ *and let* φ_i *denote the solution to*

$$(P) \quad \begin{cases} \Delta\varphi_i = 0 & \text{in} \quad \Omega \setminus \overline{B_i} \\ \dfrac{\partial\varphi_i}{\partial n} = g & \text{on} \quad \partial\Omega \\ \dfrac{\partial\varphi_i}{\partial n} = \boldsymbol{\ell}_i \cdot \mathbf{n} & \text{on} \quad \partial B_i, \qquad \displaystyle\int_{\Omega\setminus B_i} \varphi_i = 0 \end{cases}$$

Then problem (P) is identifiable in the following sense :

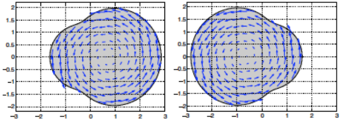
$$\varphi_1 = \varphi_2 \quad \text{on} \quad \Gamma_m \quad \Rightarrow \boldsymbol{h}_1 = \boldsymbol{h}_2 \quad \text{and} \quad \boldsymbol{\ell}_1 = \boldsymbol{\ell}_2.$$

Here, $\Omega_{ad} = \{\boldsymbol{h} \in \Omega \mid \mathrm{dist}(\boldsymbol{h}, \partial\Omega) > 1\}$.

Main issues

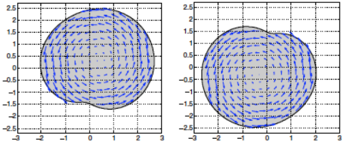
- Counterexamples shows that a same potential velocity may correspond to different positions and velocities of a same solid :

Inverse Problems **26** (2010) 095010 C Conca *et al*



(a) $\omega = 0.7$, $s_1 = (0.7, 0)$, $\lambda = 1.5$ (b) $\omega = 0.7$, $s_2 = (-0.7, 0)$, $\lambda = 1.5$

Figure 1. For both configurations, the stream function is the same. It reads $\psi(r,\theta) := \cos(2\theta)/r^2$ in polar coordinates. The holomorphic potential is $\xi(z) := \mathrm{i}/z^2$.



(a) $\omega = 0.6$, $\rho = 0.5$, $\lambda = 1.5$ and $s_1 = (\rho\cos(\pi/4), \rho\sin(\pi/4))$ (b) $\omega = 0.6$, $\rho = 0.5$, $\lambda = 1.5$ and $s_2 = (\rho\cos(-\pi/4), \rho\sin(-\pi/4))$

Figure 2. For both configurations, the stream function and the holomorphic potential are the same as in figure 1.

- However, for specific shapes (moving ellipses for instance), the problem of detection has a unique solution.
- Solids enjoying symmetries can be partially identified.
- Continuous measurements of the fluid potential over a time interval allows a continuous tracking of the solid :

Theorem 2 (Tracking)

For any solid S_0 if its position at $t = 0$ is known and continuous measurements of the potential over $[0, T]$, $T > 0$ are performed, then the configuration of the solid at any time $t \in [0, T]$ can be deduced.

Methods

Complex analysis (conformal mappings)

A moving obstacle immersed in a (stationary) viscous fluid

$$
\begin{aligned}
\operatorname{div}(\sigma(\mathbf{u},p)) &= 0 \quad \text{in} \quad F(t) = \Omega \setminus \overline{S(t)} \subset \mathbb{R}^{2,3} \\
\operatorname{div}\ \mathbf{u} &= 0 \quad \text{in} \quad F(t) \\
\mathbf{u} &= \mathbf{h}' + \boldsymbol{\omega} \times (\boldsymbol{x} - \mathbf{h}) \quad \text{on} \quad \partial S(t) \\
\mathbf{u} &= \mathbf{g} \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

$$
\int_{\partial S(t)} \sigma(\mathbf{u},p)\mathbf{n}ds = 0, \qquad \int_{\partial S(t)} (\boldsymbol{x} - \mathbf{h}) \times \sigma(\mathbf{u},p)\mathbf{n}ds = 0
$$

$$
S(0) = S_0, \qquad \mathbf{h}(0) = \mathbf{h}_0,\ \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0
$$

and for $t \in (0, T)$, where

$$
\begin{aligned}
\mathbf{h}(t) &: \quad \text{center of mass of } S(t) \\
\boldsymbol{\omega}(t) &: \quad \text{angular velocity}
\end{aligned}
$$

$\mathbf{g}$ is a given velocity satisfying the compatibility condition

$$
\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}ds = 0
$$

A simplified inverse geometrical problem
Low Reynold's number regime so inertial forces are neglected

- Assume Γ_m is a non empty open subset of $\partial\Omega$ where we can measure $\sigma(u,p)\mathbf{n}_{|\Gamma}$ at some time $t_0 > 0$.
- Is it possible to recover S_0?

- Assume Γ_m is a non empty open subset of $\partial\Omega$ where we can measure $\sigma(u,p)\mathbf{n}_{|\Gamma}$ at some time $t_0 > 0$.
- Is it possible to recover S_0?

In the sequel we seek for uniqueness of S_0. More precisely, let us take two non empty open sets $S_0^{(1)}$, $S_0^{(2)}$ and we prove that under certain assumptions on the rigid bodies and the function $\mathbf{g}$, the system is detectable unless rotation.

T. Takahashi & M. Tucsnak, *J. Math. Fluid Mech.* **6**, 2004; M. Boulakia, *J. Math. Fluid Mech.* **9**, 2007.

Theorem 3

Assume $\mathbf{g} \in \boldsymbol{H}^{3/2}(\partial\Omega)$, *$S_0$ to be a smooth non empty domain and assume* $(\mathbf{h}_0, \boldsymbol{\omega}_0) \in \mathbb{R}^N \times \mathbb{R}^N$ *(N=2 or 3) be given.*
Then there exists a maximal time $T_* > 0$ *and a unique solution*

$$\mathbf{h} \in \boldsymbol{C}^1([0, T_*); \mathbb{R}^3), \quad (\boldsymbol{h}', \boldsymbol{\omega}) \in \boldsymbol{C}([0, T_*); \mathbb{R}^3 \times \mathbb{R}^3),$$

$$(\boldsymbol{u}, p) \in \boldsymbol{C}\left([0, T_*); \boldsymbol{H}^2(F(t)) \times H^1(F(t))/\mathbb{R}\right)$$

satisfying the quasi-stationary Stokes system. Moreover one of the following alternatives holds:

- $T_* = +\infty$;
- $\lim_{t \to T_*} \operatorname{dist}(S(t), \partial\Omega) = 0$.

Preliminaries (Identifiability result)

- Let us take two smooth non empty open sets $S_0^{(1)}$, $S_0^{(2)}$
- Let us also consider $\left(\boldsymbol{h}_0^{(i)}, \boldsymbol{\omega}_0^{(i)}\right)$, $i = 1, 2$ such that the corresponding solid structures are such that

$$\overline{S_0^{(1)}} \subset \Omega \quad \text{and} \quad \overline{S_0^{(2)}} \subset \Omega$$

- Applying the above existence result, we deduce that for any $\mathbf{g} \in \boldsymbol{H}^{3/2}(\partial\Omega)$, there exists $T_*^{(1)} > 0$ $\left(\text{respectively } T_*^{(2)} > 0\right)$ and a unique solution $\left(\boldsymbol{h}^{(1)}, \boldsymbol{\omega}^{(1)}, \mathbf{u}^{(1)}, p^{(1)}\right)$ (respectively $\left(\boldsymbol{h}^{(2)}, \boldsymbol{\omega}^{(2)}, \mathbf{u}^{(2)}, p^{(2)}\right)$) in $\left[0, T_*^{(1)}\right)$ (respectively in $\left[0, T_*^{(2)}\right)$)

Theorem 4 (Identifiability)

Assume $\boldsymbol{g}$ *is not the trace of a rigid velocity on* Γ*, and that* $S_0^{(1)}$*,* $S_0^{(2)}$ *are convex. If there exists* $0 < t_0 < \min\left(T_*^{(1)}, T_*^{(2)}\right)$ *such that*

$$\sigma\left(\mathbf{u}^{(1)}(t_0), p^{(1)}(t_0)\right)\ \mathbf{n}_{|\Gamma} = \sigma\left(\mathbf{u}^{(2)}(t_0), p^{(2)}(t_0)\right)\ \mathbf{n}_{|\Gamma}$$

then there exists $R \in SO_3(\mathbb{R})$ *such that*

$$RS_0^{(1)} = S_0^{(2)}$$

and

$$\boldsymbol{h}_0^{(1)} = \boldsymbol{h}_0^{(2)}, \qquad \boldsymbol{\omega}_0^{(1)} = \boldsymbol{\omega}_0^{(2)}.$$

In particular, $T_*^{(1)} = T_*^{(2)}$ *and* $S^{(1)}(t) = S^{(2)}(t) \quad \forall t \in \left[0, T_*^{(1)}\right)$

Thank you!, Luísa