Asymptotic Analysis of the Approximate Control for Parabolic Equations with Periodic Interface

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PLAN

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7. BON ANNIVERSAIRE, LUISA !

Control Theory

- Control Theory is a branch of mathematics which aims to find a control that will lead the given state of the system in a desirable situation.
- The system: An evolution system is considered either in terms of partial or ordinary differential equations.

The Controllability Problem for P.D.E.

- Then, given the initial state at time $t = 0$, we act on the system by a control, in order to reach a desired final state at time $t = T$ (exact controllability) or to approach a desired final state (approximate controllability).
  Also, we want to act on an arbitrary zone (not everywhere..).
- The control can be the right-hand side of the equation (internal control) or the data in the boundary condition (boundary control).

The approximate controllability for parabolic problems

Due to the regularizing effect of the heat equation, one cannot reach any given L^2 state.

The approximate controllability problem

One has approximate controllability if the set of reachable final states is dense in $L^2(\Omega)$.

The variational approach

Following an idea by J.-L. Lions, the approximate control can be constructed as the solution of a related transposed (backward) problem having as final data the (unique) minimum point of a suitable functional.

J.-L. Lions, *Remarques sur la controlabilité approchée*. in Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos, octubre 1990, Grupo de Análisis Matemático Aplicado de la University of Malaga, Spain (1991), 77-87.

The approximate controllability problem for a model case

Let Ω be a connected bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ and ω a given open non-empty subset of Ω.

For the usual heat equation the problem reads

Given $w \in L^2(\Omega)$ and $\delta > 0$ find $\varphi \in L^2(\Omega)$ such that for a given $u^0 \in L^2(\Omega)$ the solution u of

$$\begin{cases} u' - div\,(A\nabla u) = \chi_\omega \varphi & in\ \Omega \times (0, T)\,, \\ u = 0 & on\ \partial\Omega \times (0, T)\,, \\ u(x, 0) = u^0 & in\ \Omega, \end{cases}$$

verifies the following approximate controllability:

$$\|u(x, T) - w\|_{L^2(\Omega)} \leq \delta.$$

In the variational approach of J.-L. Lions, φ is obtained as the solution of the following homogeneous transposed problem:

$$\begin{cases} -\varphi' - \operatorname{div}(A^0 \nabla \varphi) = 0 & \text{in } \Omega \times (0, T)\,, \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ \varphi(x, T) = \hat{\varphi}^0 & \text{in } \Omega, \end{cases}$$

the final data $\hat{\varphi}^0$ being the (unique) minimum point of the functional J_0 on $L^2(\Omega)$ given by

$$J_0(\psi^0) = \frac{1}{2} \int_0^T \int_\omega |\psi|^2 \, dx \, dt + \delta \|\psi^0\|_{L^2(\Omega)} - \int_\Omega (w - v(T))(\psi^0) \, dx,$$

where ψ is the solution of

$$\begin{cases} -\psi' - \text{div}\,(A\nabla\psi) = 0 & \text{in } \Omega \times (0, T)\,, \\ \psi = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ \psi(x, T) = \psi^0 & \text{in } \Omega \end{cases}$$

and v is the solution of the problem

$$\begin{cases} v' - \text{div}\,(A\nabla v) = 0 & \text{in } \Omega \times (0, T)\,, \\ v = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ v(x, 0) = u^0 & \text{in } \Omega. \end{cases}$$

The case of oscillating coefficients

In this case for every ε one can construct a control for the problem

Given $w_\varepsilon \in L^2(\Omega)$ and $\delta > 0$ find $\varphi_\varepsilon \in L^2(\Omega)$ such that for a given $u_\varepsilon^0 \in L^2(\Omega)$ the solution u_ε of

$$\begin{cases} {u_\varepsilon}' - \mathrm{div}\,(A^\varepsilon \nabla u_\varepsilon) = \chi_\omega \varphi_\varepsilon & \text{in } \Omega \times (0, T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u_\varepsilon(x, 0) = u_\varepsilon^0 & \text{in } \Omega, \end{cases}$$

verifies the following approximate controllability:

$$\|u_\varepsilon(x, T) - w_\varepsilon\|_{L^2(\Omega)} \leq \delta.$$

Suppose now that:

$$\begin{cases} (i) \quad u_\varepsilon^0 \to u^0 \quad \text{strongly in } L^2(\Omega), \\ (ii) \quad w_\varepsilon \to w \quad \text{strongly in } L^2(\Omega), \end{cases}$$

for some u^0 and w in $L^2(\Omega)$.

An interesting question is

Do the control and the corresponding solution of the ε-problem converge (as $\varepsilon \to 0$) to a control of the homogenized problem and to the corresponding solution, respectively?

A positive answer is given in

E. Zuazua, *Approximate Controllability for Linear Parabolic Equations with Rapidly Oscillating Coefficients*. Control Cybernet, 4 (1994), 793-801.

The case of perforated domains

Let Ω_ε a domain perforated by a set S_ε of ε-periodic holes of size ε.

Then, for every ε one can construct a control for the problem

Given $w_\varepsilon \in L^2(\Omega_\varepsilon)$ and $\delta > 0$ find $\varphi_\varepsilon \in L^2(\Omega_\varepsilon)$ such that for a given $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$ the solution u_ε of

$$\begin{cases} {u_\varepsilon}' - \operatorname{div}\,(A^\varepsilon \nabla u_\varepsilon) = \chi_\omega \varphi_\varepsilon & \text{in } \Omega_\varepsilon \times (0, T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ A^\varepsilon \nabla u_\varepsilon \cdot n = 0 & \text{on } \partial S_\varepsilon \times (0, T)\,, \\ u_\varepsilon(x, 0) = u_\varepsilon^0 & \text{in } \Omega, \end{cases}$$

verifies the following approximate controllability:

$$\|u_\varepsilon(x, T) - w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \delta.$$

Suppose that

$$\begin{cases} (i) \quad u_\varepsilon^0 \to u^0 & \text{strongly in } L^2(\Omega), \\ (ii) \quad w_\varepsilon \to w & \text{strongly in } L^2(\Omega). \end{cases}$$

Again one can give a positive answer to the previous question, see

P. Donato and A. Nabil, *Approximate Controllability of Linear Parabolic Equations in Perforated Domains*. ESAIM: Control, Optimization and Calculus of Variations, 6 (2001), 21-38.

In this case, in order to obtain the convergence result, the suitable functional in the construction of the control for the homogenized problem is

$$J(\psi^0) = \frac{1}{2}\theta \int_0^T \int_\omega |\psi|^2 \, dx \, dt + \delta\sqrt{\theta}\|\psi^0\|_{L^2(\omega)} - \theta \int_\Omega (w - v(T))\, \psi^0 \, dx,$$

where θ is the proportion of material in the reference cell.

A Heat Equation in a Composite with Interfacial Resistances

Work in collaboration with Editha C. Jose (University of the Philippines Los Baños)

We consider a more complicated case where the domain is a two-component domain, the holes being here replaced by a second material.

On the periodic interface, a jump of the solution is prescribed, which is proportional to the conormal derivative via a parameter $\gamma \in \mathbb{R}$, and a Dirichlet condition is imposed on the exterior boundary $\partial\Omega$.

This problem models the heat diffusion in a two-component composite with an imperfect contact on the interface, see for a physical justification of the model

H.S.Carslaw, J.C. Jaeger, *Conduction of Heat in Solids*. The Clarendon Press, Oxford, 1947.

We describe here the case $\gamma = 1$, which is the most interesting case since the homogenized problem is a coupled system of a P.D.E. and a O.D.E., giving rise to a memory effect.

However, our results concern all the value of $\gamma \in \mathbb{R}$.

Several questions must be addressed here :

- Can we construct an approximate control for the ε-problem ?
- Can we construct an approximate control for the homogenized coupled problem ?
- If such controls exist, do the control and the corresponding solution of ε-problem converge to a control of the homogenized problem and to the corresponding solution, respectively?
- We give here positive answers to all three questions.

The main difficulty

- To find suitable functionals for both problems, the oscillating problem and the homogenized one, which provide not only the approximate controls but also the desired convergences.

Many functionals provide control, in particular we can change the constant in the different terms of the functional and still have controllability.

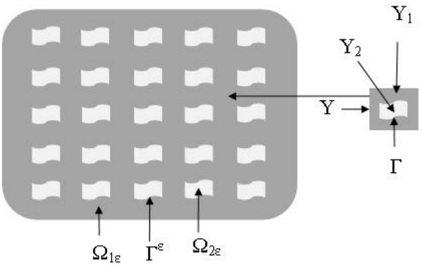
But those providing the convergence of the problem have to be carefully chosen.

Important tools

- The corrector results play an important role in the proofs.

- Unique continuation results are needed for the two problems, in particular for the homogenized coupled problem.

The Domain



Y is the reference cell where

- $Y = Y_1 \cup \overline{Y_2}$, with $\overline{Y_2} \subset Y$
- $\Gamma := \partial Y_2$ Lipschitz continuous,

The domain in $\mathbb{R}^n$:

$$\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}},$$

where

- $\Omega_{1\varepsilon}$ is a connected union of ε^{-n} periodic translated sets of εY_1,
- $\Omega_{2\varepsilon}$ is a union of ε^{-n} periodic disjoint translated sets of εY_2,
- $\Gamma_\varepsilon := \partial\Omega_{2\varepsilon}$ is the interface between the two components, with $\partial\Omega \cap \Gamma_\varepsilon = \emptyset$.

The ε-problem in the two-component domain

Consider for $\gamma \in \mathbb{R}$ the following parabolic system of equations:

$$\begin{cases} {u_{1\varepsilon}}' - \operatorname{div}(A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon}) = \chi_{\omega_{1\varepsilon}}\varphi_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0, T)\,, \\ {u_{2\varepsilon}}' - \operatorname{div}(A(\frac{x}{\varepsilon})\nabla u_{2\varepsilon}) = \chi_{\omega_{2\varepsilon}}\varphi_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0, T)\,, \\ A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A(\frac{x}{\varepsilon})\nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h(\frac{x}{\varepsilon})(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u_{1\varepsilon}(x, 0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x, 0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i = 1, 2)$, ω is a given open non-empty subset of Ω, and we set $\omega_{i\varepsilon} = \omega \cap \Omega_{i\varepsilon},\ i = 1, 2$.

Assumptions

- A is an $n \times n$ matrix field which is $Y-$ periodic, symmetric and of class $C^1(\bar{Y})$ such that for some $0 < \alpha < \beta$, one has

$$\begin{cases} (A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(y)\lambda| \leq \beta|\lambda|. \end{cases}$$

  $\forall\, \lambda \in \mathbb{R}^n$ and a.e. in Y.

- h is a Y- periodic function in $L^\infty(\Gamma)$ such that

  $\exists\ \ h_0 \in \mathbb{R} \quad \text{with} \quad 0 < h_0 < h(y), \quad y \quad \text{a.e. in } \Gamma.$

- $(U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$,

- $(\varphi_{1\varepsilon}, \varphi_{2\varepsilon}) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$.

Remark Observe that for any $v \in L^2(\Omega)$

$$v = v\chi_{\Omega_{1\varepsilon}} + v\chi_{\Omega_{2\varepsilon}}.$$

Then the map

$$\Phi : v \in L^2(\Omega) \to (v|_{\Omega_{1\varepsilon}}, v|_{\Omega_{2\varepsilon}}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$$

is a bijective isometry since

$$\|v\|^2_{L^2(\Omega)} = \|v\|^2_{L^2(\Omega_{1\varepsilon})} + \|v\|^2_{L^2(\Omega_{2\varepsilon})}, \quad \text{for every } v \in L^2(\Omega).$$

⋆ In the sequel, when needed we identify $v \in L^2(\Omega)$ with $(v|_{\Omega_{1\varepsilon}}, v|_{\Omega_{2\varepsilon}}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$.

The variational formulation

We set

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \qquad h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right).$$

and we introduce the functional spaces

$$V^\varepsilon := \{v_1 \in H^1(\Omega_{1\varepsilon}) \mid v_1 = 0 \text{ on } \partial\Omega\}$$

equipped with the norm

$$\|v_1\|_{V^\varepsilon} := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}$$

and

$$W^\varepsilon \doteq \begin{cases} v = (v_1, v_2) \in L^2(0,T;V^\varepsilon) \times L^2(0,T;H^1(\Omega_{2\varepsilon})) \text{ s.t.} \\ v' \in L^2(0,T;(V^\varepsilon)') \times L^2(0,T;(H^1(\Omega_{2\varepsilon}))')\}, \end{cases}$$

equipped with the norm

$$\|v\|_{W^\varepsilon} = \|v_1\|_{L^2(0,T;V^\varepsilon)} + \|v_2\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))} + \|v_1'\|_{L^2(0,T;(V^\varepsilon)')} + \|v_2'\|_{L^2(0,T;(H^1(\Omega_{2\varepsilon}))')}.$$

Then, the variational formulation associated to the problem is

$$
\begin{cases}
\text{Find } u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\[2ex]
\displaystyle\int_0^T \langle u_{1\varepsilon}', v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} \, \mathrm{d}t + \int_0^T \langle u_{2\varepsilon}', v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \, \mathrm{d}t \\[2ex]
\displaystyle + \int_0^T \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla v_1 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla v_2 \, \mathrm{d}x \, \mathrm{d}t \\[2ex]
\displaystyle + \varepsilon^\gamma \int_0^T \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2) \, \mathrm{d}\sigma_x \, \mathrm{d}t \\[2ex]
\displaystyle = \int_0^T \int_{\omega_{1\varepsilon}} \varphi_{1\varepsilon} v_1 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\omega_{2\varepsilon}} \varphi_{2\varepsilon} v_2 \, \mathrm{d}x \, \mathrm{d}t \\[2ex]
\text{for every } (v_1, v_2) \in L^2(0, T; V^\varepsilon) \times L^2(0, T; H^1(\Omega_{2\varepsilon})) \\[3ex]
u_{1\varepsilon}(x, 0) = U_{1\varepsilon}^0 \text{ in } \Omega_{1\varepsilon} \ \text{ and } \ u_{2\varepsilon}(x, 0) = U_{2\varepsilon}^0 \text{ in } \Omega_{2\varepsilon}.
\end{cases}
$$

The related Controllability Problem

Given $w_{i\varepsilon} \in L^2(\Omega_{i\varepsilon}), i = 1, 2$ $\delta_1 > 0$ and $\delta_2 > 0$, find a control $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ such that the solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ of the above problem verifies the estimates

$$\begin{cases} (i) \quad \|u_{1\varepsilon}(T) - w_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \le \delta_1 \\ \\ (ii) \quad \|u_{2\varepsilon}(T) - w_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \le \delta_2. \end{cases}$$

Some References

- E. Jose, *Homogenization of a Parabolic Problem with an Imperfect Interface*. Rev. Roumaine Math. Pures Appl., 54 (2009) (3), 189-222.
- P. Donato, E. Jose, *Corrector Results for a Parabolic Problem with a Memory Effect*. ESAIM: M2AN 44 (2010), 421-454.
- P. Donato, E. Jose, *Asymptotic behavior of the approximate controls for parabolic equations with interfacial contact resistance*. ESAIM: Control, Optimisation and Calculus of Variations, 21 (1), (2015), 138-164, DOI 10.1051/cocv/2014029.
- P. Donato, E. Jose, *Approximate Controllability of a Parabolic System with Imperfect Interfaces*, to appear in Philippine Journal of Sciences.
- F. Ammar Khodja, personal communication, Appendix.

The Variational Approach to the Controllability Problem

Let $(w_{1\varepsilon}, w_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ and $\varphi^0 \in L^2(\Omega)$ and define the functional J_ε by

$$
\begin{aligned}
J_\varepsilon(\varphi^0) =& \frac{1}{2}\left(\int_0^T \int_{\omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 \, dx \, dt \int_0^T \int_{\omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 \, dx \, dt\right) + \\
& + \delta_1 \|\varphi^0\|_{L^2(\Omega_{1\varepsilon})} + \delta_2 \|\varphi^0\|_{L^2(\Omega_{2\varepsilon})} \\
& - \int_{\Omega_{1\varepsilon}} (w_{1\varepsilon} - v_{1\varepsilon}(T))\varphi^0 \, dx - \int_{\Omega_{2\varepsilon}} (w_{2\varepsilon} - v_{2\varepsilon}(T))\varphi^0 \, dx,
\end{aligned}
$$

where $\theta_i = \dfrac{|Y_i|}{|Y|}$ for $i = 1, 2$.

$\star$ Observe that $\chi_{\Omega_{i\varepsilon}}$ converges to θ_i in L^∞ only weakly *. Then, one difficulty in this study is that the L^2-weak convergence of a function v_ε to some v do not imply the convergence of $\chi_{\Omega_{i\varepsilon}} v_\varepsilon$ to $\theta_i v$.

In the functional defined above, $\varphi_\varepsilon = (\varphi_{1\varepsilon}, \varphi_{2\varepsilon})$ is the solution of the transposed problem of the system given by

$$\begin{cases} -\varphi_{1\varepsilon}' - \operatorname{div}(A^\varepsilon \nabla \varphi_{1\varepsilon}) = 0 & \text{in } \Omega_{1\varepsilon} \times (0, T)\,, \\ -\varphi_{2\varepsilon}' - \operatorname{div}(A^\varepsilon \nabla \varphi_{2\varepsilon}) = 0 & \text{in } \Omega_{2\varepsilon} \times (0, T)\,, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \varphi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h_\varepsilon (\varphi_{1\varepsilon} - \varphi_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ \varphi_{1\varepsilon}(x, T) = \varphi^0|_{\Omega_{1\varepsilon}} & \text{in } \Omega_{1\varepsilon}, \\ \varphi_{2\varepsilon}(x, T) = \varphi^0|_{\Omega_{2\varepsilon}} & \text{in } \Omega_{2\varepsilon}. \end{cases}$$

On the other hand, $v_\varepsilon = (v_{1\varepsilon}, v_{2\varepsilon})$ is the solution of the auxiliary problem

$$
\begin{cases}
{v_{1\varepsilon}}' - \operatorname{div}(A^\varepsilon \nabla v_{1\varepsilon}) = 0 & \text{in } \Omega_{1\varepsilon} \times (0, T)\,, \\
{v_{2\varepsilon}}' - \operatorname{div}(A^\varepsilon \nabla v_{2\varepsilon}) = 0 & \text{in } \Omega_{2\varepsilon} \times (0, T)\,, \\
A^\varepsilon \nabla v_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla v_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\
A^\varepsilon \nabla v_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h_\varepsilon (v_{1\varepsilon} - v_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\
v_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T)\,, \\
v_{1\varepsilon}(x, 0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\
v_{2\varepsilon}(x, 0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon},
\end{cases}
$$

where n_i is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i = 1, 2)$ and $(U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$.

The Controllability Result for Fixed ε

Theorem 1 [D-Jose] Let $T > 0$, $\delta_1 > 0$, $\delta_2 > 0$ be given real numbers and U^0_ε be in $L^2(\Omega)$. Fix $w_\varepsilon = (w_{1\varepsilon}, w_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$.

Let $\widehat{\varphi}^0_\varepsilon$ be the unique minimum point of the functional J_ε and $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ the solution of the transposed problem with final data $\widehat{\varphi}^0_\varepsilon$.

Then the solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ of the following system:

$$\begin{cases} {u_{1\varepsilon}}' - \mathrm{div}(A^\varepsilon \nabla u_{1\varepsilon}) = \chi_{\omega_{1\varepsilon}} \widehat{\varphi}_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0, T)\,, \\ {u_{2\varepsilon}}' - \mathrm{div}(A^\varepsilon \nabla u_{2\varepsilon}) = \chi_{\omega_{2\varepsilon}} \widehat{\varphi}_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0, T)\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_2 & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_1 = -\varepsilon^\gamma h_\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0, T)\,, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u_{1\varepsilon}(x, 0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x, 0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases}$$

satisfies the following estimate:

$$\begin{cases} (i)\ \|u_{1\varepsilon}(T) - w_\varepsilon\|_{L^2(\Omega_{1\varepsilon})} \leq \delta_1 \\ (ii)\ \|u_{2\varepsilon}(T) - w_\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \leq \delta_2. \end{cases}$$

Sketch of the proof

Existence of the minimum

- By standard arguments one can prove that the functional J_ε is continuous and strictly convex.
- Then, for every fixed ε, we prove the coerciveness, i.e.

$$\liminf_{\|\varphi_\varepsilon^0\|_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(\varphi_\varepsilon^0)}{\|\varphi_\varepsilon^0\|_{L^2(\Omega)}} \geq \min\{\delta_1,\ \delta_2\}.$$

$\star$To do that it is essential to use the unique-continuation property of Saut-Scheurer.

Proof of the control result

- Let $\psi_\varepsilon^0 \in L^2(\Omega)$ and $i = 1, 2$. If $\widehat{\varphi}_\varepsilon^0$ is the minimum point of the functional J_ε we have

$$\begin{cases} \left| \displaystyle\sum_{i=1}^{2} \left(\int_0^T \int_{\omega_i} \widehat{\varphi}_{i\varepsilon} \psi_{i\varepsilon} \, dx \, dt - \int_{\Omega_{i\varepsilon}} (w_\varepsilon - v_{i\varepsilon}(T)) \psi_\varepsilon^0 \, dx \right) \right| \\ \leq \delta_1 \|\psi_\varepsilon^0\|_{L^2(\Omega_{1\varepsilon})} + \delta_2 \|\psi_\varepsilon^0\|_{L^2(\Omega_{2\varepsilon})}, \end{cases}$$

  where $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ is the solution of the transposed problem with the corresponding final data $\widehat{\varphi}_\varepsilon^0$.

- Then we prove that

$$\left| \int_{\Omega_{i\varepsilon}} (u_{i\varepsilon}(T) - w_\varepsilon) \psi_\varepsilon^0 \, dx \right| \leq \delta_i \|\psi_\varepsilon^0\|_{L^2(\Omega_{1\varepsilon})}, \; \forall \psi_\varepsilon^0 \in L^2(\Omega_{i\varepsilon}), \quad i = 1, 2,$$

  which implies the controllability result.

A review of the homogenization and correctors results

$\star$ From now on, we consider $\gamma = 1$, the most interesting case.

The homogenization result [Jose]

Let A^ε and h^ε be as before and $z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon})$ be the solution of

$$\begin{cases} {z_{i\varepsilon}}' - \operatorname{div}(A^\varepsilon \nabla z_{i\varepsilon}) = g_{i\varepsilon} & \text{in } \Omega_{i\varepsilon} \times (0,T)\,,\ i = 1,2, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla z_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon\, h^\varepsilon (z_{1\varepsilon} - z_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ z_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ z_{i\varepsilon}(x,0) = Z^0_\varepsilon|_{\Omega_{i\varepsilon}} & \text{in } \Omega_{i\varepsilon},\ i = 1,2, \end{cases}$$

where $Z^0_\varepsilon \in L^2(\Omega)$ and $(g_{1\varepsilon}, g_{2\varepsilon}) \in [L^2(0,T; L^2(\Omega))]^2$. Suppose that

$$\begin{cases} (i) \quad (\chi_{\Omega_{1\varepsilon}} Z^0_\varepsilon, \chi_{\Omega_{2\varepsilon}} Z^0_\varepsilon) \rightharpoonup (\theta_1 Z^0_1, \theta_2 Z^0_2) & \text{weakly in } [L^2(\Omega)]^2, \\ (ii) \quad (\chi_{\Omega_{1\varepsilon}} g_{1\varepsilon}, \chi_{\Omega_{1\varepsilon}} g_{2\varepsilon}) \rightharpoonup (\theta_1 g_1, \theta_2 g_2) & \text{weakly in } [L^2(0,T; L^2(\Omega))]^2, \end{cases}$$

Then there exists a linear continuous extension operator

$P_1^\varepsilon \in \mathcal{L}(L^2(0,T;V^\varepsilon); L^2(0,T;H_0^1(\Omega))) \cap \mathcal{L}(L^2(0,T;L^2(\Omega_{1\varepsilon})); L^2(0,T;L^2(\Omega)))$ such that

$$\begin{cases} (i) \quad P_1^\varepsilon z_{1\varepsilon} \rightharpoonup z_1 & \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ (ii) \quad \widetilde{z_{1\varepsilon}} \rightharpoonup \theta_1 z_1 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (iii) \quad \widetilde{z_{2\varepsilon}} \rightharpoonup z_2 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (iv) \quad \varepsilon^{\frac{1}{2}} \|z_{1\varepsilon} - z_{2\varepsilon}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} < c, \end{cases}$$

where $\sim$ denotes the zero extension to the whole of Ω.
Furthermore,

$$\begin{cases} (i) \quad A^\varepsilon \widetilde{\nabla z_{1\varepsilon}} \rightharpoonup A^0 \nabla z_1 & \text{weakly in } L^2(0,T;[L^2(\Omega)]^n), \\ (ii) \quad A^\varepsilon \widetilde{\nabla z_{2\varepsilon}} \rightharpoonup 0 & \text{weakly in } L^2(0,T;[L^2(\Omega)]^n), \end{cases}$$

where $A^0\lambda := m_Y(A\widetilde{\widehat{w}_\lambda})$, the function $\widehat{w}_\lambda \in H^1(Y_1)$ being for any $\lambda \in \mathbb{R}^n$, the unique solution of the problem

$$\begin{cases} -\text{div}\,(A\nabla \widehat{w}_\lambda) = 0 & \text{in } Y_1, \\ (A\nabla \widehat{w}_\lambda)\cdot n_1 = 0 & \text{in } \Gamma, \\ \widehat{w}_\lambda - \lambda\cdot y \;\; Y\text{-periodic and } \; m_{Y_1}(\widehat{w}_\lambda - \lambda\cdot y) = 0. \end{cases}$$

The pair $(z_1, z_2) \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \times C^0([0,T]; L^2(\Omega))$ with $z_1' \in L^2(0,T; \mathrm{H}^{-1}(\Omega))$ is the unique solution of the homogenized coupled system

$$\begin{cases} \theta_1 z_1' - \operatorname{div}(A^0 \nabla z_1) + c_h(\theta_2 z_1 - z_2) = \theta_1 g_1 & \text{in } \Omega \times (0,T), \\ z_2' - c_h(\theta_2 z_1 - z_2) = \theta_2 g_2 & \text{in } \Omega \times (0,T), \\ z_1 = 0 & \text{on } \partial\Omega \times (0,T), \\ z_1(0) = Z_1^0, \ z_2(0) = \theta_2 Z_2^0 & \text{in } \Omega, \end{cases}$$

where $c_h = \dfrac{1}{|Y_2|} \displaystyle\int_\Gamma h(y)\, d\sigma_y$.

$\star$ This result can be interpretated as a memory effect. Indeed, solving the EDO and replacing in the PDE gives a PDE with a memory term.

The corrector result [D-Jose]

Under the assumption of the homogenization theorem, suppose further that for $Z_\varepsilon^0 \in L^2(\Omega)$ and $g_{i\varepsilon} \in L^2(0,T;L^2(\Omega))$, $i=1,2$ one has

$$(\chi_{\Omega_{1\varepsilon}} Z_\varepsilon^0, \chi_{\Omega_{2\varepsilon}} Z_\varepsilon^0) \rightharpoonup (\theta_1 Z_1^0, \theta_2 Z_2^0) \quad \text{weakly in } [L^2(\Omega)]^2$$

and

$$\begin{cases} (i) \quad g_{i\varepsilon} \to g_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \\ (ii) \quad \|Z_\varepsilon^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|Z_\varepsilon^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|Z_1^0\|^2_{L^2(\Omega)} + \theta_2 \|Z_2^0\|^2_{L^2(\Omega)}. \end{cases}$$

Remark In particular, assumption (i) holds if for $i=1,2$, $g_{i\varepsilon} = g_\varepsilon|_{\Omega_{i\varepsilon}}$ and $g_\varepsilon \to g$ strongly in $L^2(0,T;L^2(\Omega))$.

On the other hand, the assumptions on the initial data hold if for $i=1,2$, one has for instance $\chi_{\Omega_{i\varepsilon}} Z_\varepsilon^0 = Z_{i\varepsilon}^0|_{\Omega_{i\varepsilon}}$ for some $Z_{i\varepsilon}^0 \in L^2(\Omega)$ such that $Z_{i\varepsilon}^0 \to Z_i^0$ strongly in $L^2(\Omega)$ (as it will be true in the control problem).

If $(e_j)_{j=1,\dots,n}$ is the canonical basis of $\mathbb{R}^n$ and $\widehat{w}_j$ is the solution of the cell problem written for $\lambda = e_j$, $j = 1, \dots, n$, let $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le n}$ be the corrector matrix defined, for $i, j = 1, \dots, n$, by

$$C_{ij}(y) := \frac{\partial \widehat{w}_j}{\partial y_i}(y), \quad \text{a.e. on } Y_1, \qquad C^\varepsilon_{ij}(x) = \widetilde{C_{ij}}\left(\frac{x}{\varepsilon}\right) \text{a.e. on } \Omega.$$

Assuming that Γ is of class $\mathcal{C}^2$, the following corrector results hold true:

$$\begin{cases} (i) & \lim\limits_{\varepsilon\to 0} \|z_{1\varepsilon} - z_1\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (ii) & \lim\limits_{\varepsilon\to 0} \|z_{2\varepsilon} - \theta_2^{-1} z_2\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (iii) & \lim\limits_{\varepsilon\to 0} \|\nabla z_{1\varepsilon} - C^\varepsilon \nabla z_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iv) & \lim\limits_{\varepsilon\to 0} \|\nabla z_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0. \end{cases}$$

Construction of the Control for the Homogenized Problem

Let $T > 0$, $\delta_1 > 0$, $\delta_2 > 0$ be given, w be given in $L^2(\Omega)$ and U_1^0 and U_2^0 be in $L^2(\Omega)$.

For a given $w \in L^2(\Omega)$, we define the functional J_0 on $L^2(\Omega) \times L^2(\Omega)$ by

$$J_0(\Phi^0, \Psi^0) = \frac{1}{2}\theta_1 \int_0^T \int_\omega |\varphi_1|^2 \, dx \, dt + \frac{1}{2}\theta_2^{-1} \int_0^T \int_\omega |\varphi_2|^2 \, dx \, dt$$

$$+\delta_1 \sqrt{\theta_1} \|\Phi^0\|_{L^2(\Omega)} + \delta_2 \sqrt{\theta_2} \|\Psi^0\|_{L^2(\Omega)}$$

$$- \theta_1 \int_\Omega (w - v_1(T)) \, \Phi^0 \, dx - \theta_2 \int_\Omega \left(w - \theta_2^{-1} v_2(T)\right) \Psi^0 \, dx,$$

where (φ_1, φ_2) is the solution of the following homogeneous transposed problem:

$$\begin{cases} -\theta_1 {\varphi_1}' - \mathrm{div}(A^0 \nabla \varphi_1) + c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times (0, T)\,, \\ -{\varphi_2}' - c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times (0, T)\,, \\ \varphi_1 = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ \varphi_1(x, T) = \Phi^0, \quad \varphi_2(x, T) = \theta_2 \Psi^0 & \text{in } \Omega \end{cases}$$

and (v_1, v_2) is the solution of the problem

$$\begin{cases} \theta_1 {v_1}' - \mathrm{div}(A^0 \nabla v_1) + c_h(\theta_2 v_1 - v_2) = 0 & \text{in } \Omega \times (0, T)\,, \\ {v_2}' - c_h(\theta_2 v_1 - v_2) = 0 & \text{in } \Omega \times (0, T)\,, \\ v_1 = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ v_1(x, 0) = U_1^0, \quad v_2(x, 0) = \theta_2 U_2^0 & \text{in } \Omega. \end{cases}$$

The Controllability Result for the coupled system

Theorem 2 [D-Jose] Let $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ be the unique minimum point of the functional J_0 and $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ the solution of (36) with final data $(\widehat{\Phi}^0, \theta_2\widehat{\Psi}^0)$. Then if (u_1, u_2) is the solution of

$$\begin{cases} \theta_1 {u_1}' - \mathrm{div}(A^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \chi_\omega \theta_1 \widehat{\varphi}_1 & \text{in } \Omega \times (0, T)\,, \\ {u_2}' - c_h(\theta_2 u_1 - u_2) = \chi_\omega \widehat{\varphi}_2 & \text{in } \Omega \times (0, T)\,, \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u_1(x,0) = U_1^0, \quad u_2(x,0) = \theta_2 U_2^0 & \text{in } \Omega, \end{cases}$$

we have the following approximate controllability:

$$\|\theta_1 u_1(x, T) + u_2(x, T) - w\|_{L^2(\Omega)} \leq \delta_1 \sqrt{\theta_1} + \delta_2 \sqrt{\theta_2}.$$

Sketch of the proof

- By standard arguments one can prove that the functional J_0 is continuous and strictly convex.
- Then, we have to prove the coerciveness, i.e. as before, that for any sequence $\{(\Phi_n^0, \Psi_n^0)\}$ in $[L^2(\Omega)]^2$ such that $\|(\Phi_n^0, \Psi_n^0)\|_{[L^2(\Omega)]^2} \to \infty$, one has

$$\liminf_{n\to\infty} \frac{J_0(\Phi_n^0, \Psi_n^0)}{\|(\Phi_n^0, \Psi_n^0)\|_{[L^2(\Omega)]^2}} \geq \min\{\delta_1\sqrt{\theta_1},\ \delta_2\sqrt{\theta_2}\}.$$

- Here the proof is more delicate, in particular the unique-continuation property of Saut-Scheurer cannot be applied to the EDP-EDO coupled system.
  We use a non trivial result proved "ad hoc" by F. Ammar Khodja (University of Besançon), written in Appendix of our paper.
- Then, J_0 has a unique minimum point.
- Again, we are able to derive the controllability result.

Asymptotic Behaviour of the Control Problem

Theorem 3 [D-Jose] Suppose that $T, \delta_1, \delta_2 > 0$ and that Γ is of class $\mathcal{C}^2$.
Let w_ε and U_ε^0 be given in $L^2(\Omega)$.
Let $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ and $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ the related solution and the approximates control given by Theorem 2, respectively.
For $\{w_\varepsilon\}_\varepsilon \subset L^2(\Omega)$ and $\{U_\varepsilon^0\}_\varepsilon \subset L^2(\Omega)$, we suppose that for some $U_i^0, i = 1, 2$ and w in $L^2(\Omega)$, they satisfy the following assumptions:

$$\begin{cases} (i) \quad \chi_{\Omega_{i\varepsilon}} U_\varepsilon^0 \rightharpoonup \theta_i U_i^0 \quad \text{weakly in } L^2(\Omega), \\ (ii) \quad \|U_\varepsilon^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|U_\varepsilon^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|U_1^0\|^2_{L^2(\Omega)} + \theta_2 \|U_2^0\|^2_{L^2(\Omega)}, \\ (iii) \quad w_\varepsilon \to w \quad \text{strongly in } L^2(\Omega). \end{cases}$$

(Recall that in particular we can suppose that $\chi_{\Omega_{i\varepsilon}} U_\varepsilon^0 = U_{i\varepsilon}^0|_{\Omega_{i\varepsilon}}$ with $U_{i\varepsilon}^0 \to U_i^0$ strongly in $L^2(\Omega)$.)

Then as $\varepsilon \to 0$, one has

$$\begin{cases} (i) \quad \chi_{\omega_{1\varepsilon}} \widetilde{\widehat{\varphi}_{1\varepsilon}} \rightharpoonup \chi_\omega \theta_1 \widehat{\varphi}_1 & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ (ii) \quad \chi_{\omega_{2\varepsilon}} \widetilde{\widehat{\varphi}_{2\varepsilon}} \rightharpoonup \chi_\omega \widehat{\varphi}_2 & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ (iii) \quad (\chi_{\Omega_{1\varepsilon}} \widehat{\varphi}^0_\varepsilon, \chi_{\Omega_{2\varepsilon}} \widehat{\varphi}^0_\varepsilon) \rightharpoonup (\theta_1 \widehat{\Phi}^0, \theta_2 \widehat{\Psi}^0) & \text{weakly in } [L^2(\Omega)]^2, \end{cases}$$

where $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ is the solution of the trasposed problem with final data $(\widehat{\Phi}^0, \theta_2 \widehat{\Psi}^0)$ and $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ is the unique minimum point of the functional J_0. Moreover,

$$\begin{cases} (i) \quad P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ (ii) \quad \widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (iii) \quad \widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \end{cases}$$

where the couple (u_1, u_2) satisfies

$$\begin{cases} \theta_1 {u_1}' - \operatorname{div}(A^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 \chi_\omega \widehat{\varphi}_1 & \text{in } \Omega \times (0, T)\,, \\ {u_2}' - c_h(\theta_2 u_1 - u_2) = \chi_\omega \widehat{\varphi}_2 & \text{in } \Omega \times (0, T)\,, \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u_1(x, 0) = U_1^0, \quad u_2(x, 0) = \theta_2 U_2^0 & \text{in } \Omega. \end{cases}$$

The couple $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ is an approximate control for the homogenized problem (41) corresponding to w and the constants δ_1 and δ_2, that is

$$\|\theta_1 u_1(x, T) + u_2(x, T) - w\|_{L^2(\Omega)} \leq \delta_1 \sqrt{\theta_1} + \delta_2 \sqrt{\theta_2}.$$

Sketch of the proof

The proof is long and lies on several propositions.

Proposition 1 The functionals J_ε are uniformly coercive, that is,

$$\liminf_{\substack{\|\varphi_\varepsilon^0\|_{L^2(\Omega)} \to \infty \\ \varepsilon \to 0}} \frac{J_\varepsilon(\varphi_\varepsilon^0)}{\|\varphi_\varepsilon^0\|_{L^2(\Omega)}} \geq \min\{\delta_1,\ \delta_2\}.$$

Corollary Let $\widehat{\varphi}_\varepsilon^0$ the unique minimum point of J_ε. Then, there exists a constant C independent of ε such that

$$\|\widehat{\varphi}_\varepsilon^0\|_{L^2(\Omega)} \leq C.$$

Hence, there exists $(\xi^0, \nu^0) \in [L^2(\Omega)]^2$ such that (up to a subsequence)

$$\left(\chi_{\Omega_{1\varepsilon}} \widehat{\varphi}_\varepsilon^0, \chi_{\Omega_{2\varepsilon}} \widehat{\varphi}_\varepsilon^0\right) \rightharpoonup \left(\theta_1 \xi^0, \theta_2 \nu^0\right) \quad \text{weakly in } [L^2(\Omega)]^2.$$

Then, we prove the following two essential results (in the spirit of the Γ-convergence), whose proof is rather technical:

Proposition 2 The functional J_ε satisfies

$$\lim_{\varepsilon \to 0} J_\varepsilon \left(\chi_{\Omega_{1\varepsilon}} \Phi^0 + \chi_{\Omega_{2\varepsilon}} \Psi^0 \right) = J_0 \left(\Phi^0, \Psi^0 \right),$$

for every (Φ^0, Ψ^0) in $[L^2(\Omega)]^2$.

Remark In the proof we use the corrector results for the transposed problem with final data $\chi_{\Omega_{1\varepsilon}} \Phi^0 + \chi_{\Omega_{2\varepsilon}} \Psi^0$.

Proposition 3 For any sequence $\{\psi_\varepsilon^0\}_\varepsilon \subset L^2(\Omega)$ such that as $\varepsilon \to 0$,

$$\left(\chi_{\Omega_{1\varepsilon}} \psi_\varepsilon^0, \chi_{\Omega_{2\varepsilon}} \psi_\varepsilon^0 \right) \rightharpoonup (\theta_1 \Phi^0, \theta_2 \Psi^0) \quad \text{weakly in } [L^2(\Omega)]^2,$$

for some (Φ^0, Ψ^0) in $[L^2(\Omega)]^2$, we have

$$\liminf_{\varepsilon \to 0} J_\varepsilon(\psi_\varepsilon^0) \geq J_0(\Phi^0, \Psi^0).$$

This allows to prove

Theorem 4 Let U^0 and w be given in $L^2(\Omega)$. Let $\widehat{\varphi}^0_\varepsilon$ be the minimum point of J_ε, and $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ the unique minimum point of the functional J_0. Then, as $\varepsilon \to 0$,

$$\left(\chi_{\Omega_{1\varepsilon}} \widehat{\varphi}^0_\varepsilon, \chi_{\Omega_{2\varepsilon}} \widehat{\varphi}^0_\varepsilon\right) \rightharpoonup \left(\theta_1 \widehat{\Phi}^0, \theta_2 \widehat{\Psi}^0\right) \quad \text{weakly in } [L^2(\Omega)]^2.$$

Proof From the Corollary and Proposition 3 we have

$$J_0(\xi^0, \nu^0) \leq \liminf_{\varepsilon \to 0} J_\varepsilon(\widehat{\varphi}^0_\varepsilon).$$

Since $\widehat{\varphi}^0_\varepsilon$ is the minimum point of the functional J_ε, for any $(\Phi^0, \Psi^0) \in [L^2(\Omega)]^2$, using Proposition 2 we have,

$$\limsup_{\varepsilon \to 0} J_\varepsilon(\widehat{\varphi}^0_\varepsilon) \leq \limsup J_\varepsilon\left(\chi_{\Omega_{1\varepsilon}} \Phi^0 + \chi_{\Omega_{2\varepsilon}} \Psi^0\right) = J_0(\Phi^0, \Psi^0). \tag{1}$$

Then, we get $(\xi^0, \nu^0) = (\widehat{\Phi}^0, \widehat{\Psi}^0)$ where $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ is the unique minimum point of the functional J_0 and consequently, the whole sequence in the Corollary converges.

Proof of Theorem 3

Once proved Theorem 4, using the homogenization and correctors results we can pass to the limit in all the problems which complete the proof !


Happy BIRTHDAY
LUISA !!