A Chromaticity-Brightness Model for Color Images Denoising

in collaboration with Luisa Mascarenhas and Rita Ferreira

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Digital Image as a Mathematical Object

Digital image – obtained from the analogue image (physical, real image) by *sampling* and *quantization*

The digital camera superimposes a regular grid on an analogue image and assigns a value, e.g., the mean brightness in this field, to each grid element (pixels)

The image content is described by
• grey values in each pixel – scalar values ranging between 0 (black) and 255 (white)
or
• colour values prescribed in each pixel –(r,g,b), where each channel r,g,b represents the red, green, and blue component of the colour and ranges from 0 to 255.

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Size of Digital Images

Typical sizes of digital images range from

– 2000×2000 pixels in images taken with simple digital cameras

– to 10000×10000 pixels in images taken with high-resolution cameras used by professional photographers.

The size of images in medical imaging applications depends on the task at hand. PET (Positron emission tomography) produces three dimensional image data, where a full-length body scan has a typical size of $175 \times 175 \times 500$ pixels.

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Digital Image - a Mathematical Function

Mathematical representation of a digital image (sampled and quantized version of an *analog*):
image function u defined on a two dimensional (in general rectangular) image domain $\Omega := (a, b) \times (c, d)$ (the grid)

$$u : \Omega \to \mathbb{R}$$

or

$$u : \Omega \to \mathbb{R}^3$$

N. B. The domain Ω could be three dimensional, e.g. videos, 3D medical imaging

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The next figure visualizes the connection between the digital image and its image function.

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Digital Image and its Image Function

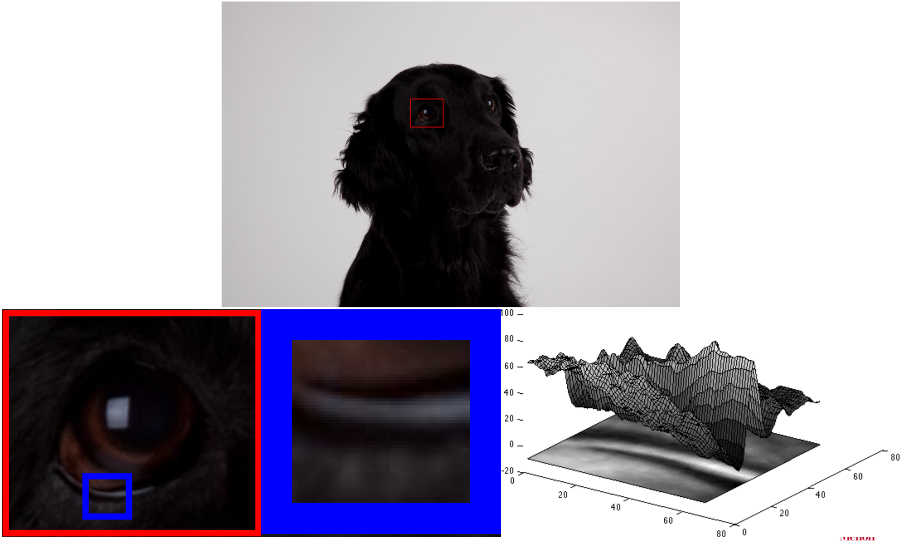

Figure: Digital image versus image function: Gradually zooming in until we are at level where the image pixels are visible (blue framed detail), the image function of the red channel $u(x, y, r)$ of the digital photograph is plotted as the height (the value for red) over the (x, y)–plane.

Digital Image as a Mathematical Object

Since the image function is a mathematical object we can treat it as such and apply mathematical operations to it!

These mathematical operations–**image processing techniques**–include:

- statistical methods
- morphological operations
- solving a partial differential equation for the image function

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$BV(\Omega)$ – especially suited for images since an element in BV can be discontinuous and hence the representation of image edges is possible

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Some Challenges . . .

- Deblurring
- Segmentation
- Denoising
- Inpainting
- Recolorization

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Deblurring with the TV Model

Compute u for given $f = Ku$, where K is a linear and bounded operator, e.g. a convolution with a Gaussian kernel

$$\alpha|Du|(\Omega) + \frac{1}{2}\|Ku - f\|^2_{L^2(\Omega)}.$$

Figure: Blurred and deblurred image using total variation regularisation.

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Segmentation

Image segmentation aims to segment one or more objects of interest in an image, also under the presence of noise and blur.

Geodesic active contour segmentation computes the boundary of an object as the zero-level set of a stationary solution of

$$\varphi_t = |\nabla\varphi| \operatorname{div}\left(g(|\nabla f|)\, \frac{\nabla\varphi}{|\nabla\varphi|} \right),$$

g ... edge detector function, e.g., $g(s) = \frac{1}{1+s^2}$

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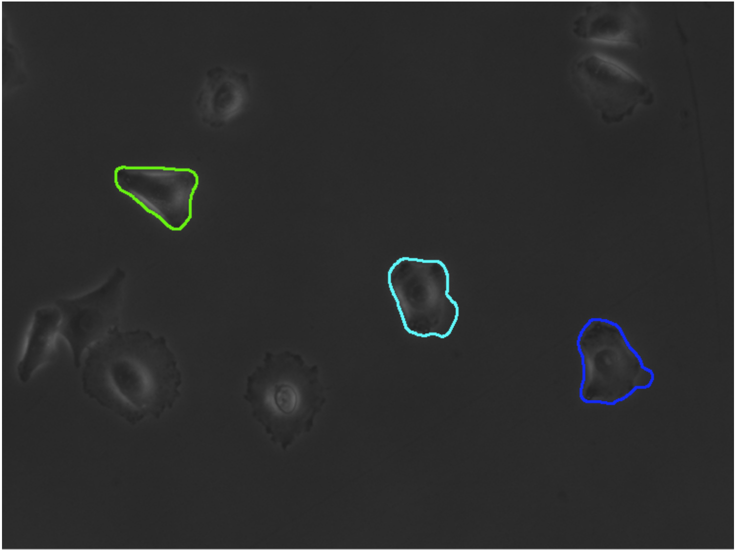
Segmentation Using an Extended Version of the Geodesic Active Contour Method



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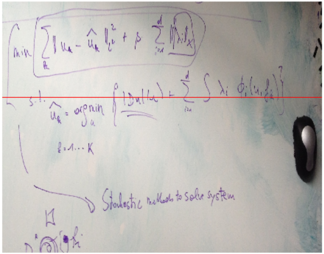
Denoising

In most acquisition processes for digital images data **wrong information is added to the image**. Even modern cameras which are able to acquire high–resolution images produce **noisy outputs**



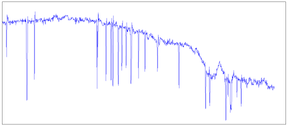


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Figure: Bad lighting conditions may result into noisy image. First: A digital photo which has been acquired under too little light. Second: Plot of the values of the red channel along the one-dimensional slice marked in red in the photograph.

More on Denoising

For the human eye, noise is an easy problem to cope with. If the noise is not too strong we are still able to analyze an image for its contents.

However, for the computer this is not the case. This is important when aiming for the automated analysis of an image.

Total variation (TV) denoising – to recover u from $f = u + n$ (for additive noise n) by minimizing

$$\alpha|Du|(\Omega) + \frac{1}{2}\|u - f\|^2_{L^2(\Omega)}.$$

$f \in L^2(\Omega)$ ... noisy image
u_α minimizer the above functional for a fixed positive α... denoised image

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Denoising with the TV Model

• Linear filtering (e.g. Gaussian) – lets the image evolve along the heat equation up to a certain time. Smoothes the image everywhere with the same strength

By contrast,

• TV denoising is a nonlinear denoising filter that smoothes more in homogeneous areas of the image and less at image edges

Formally a minimizer of the TV functional solves

$$-\alpha \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + u - f = 0,$$

More diffusion where the image gradient is small and less diffusion where it is large

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Denoising with the TV Model

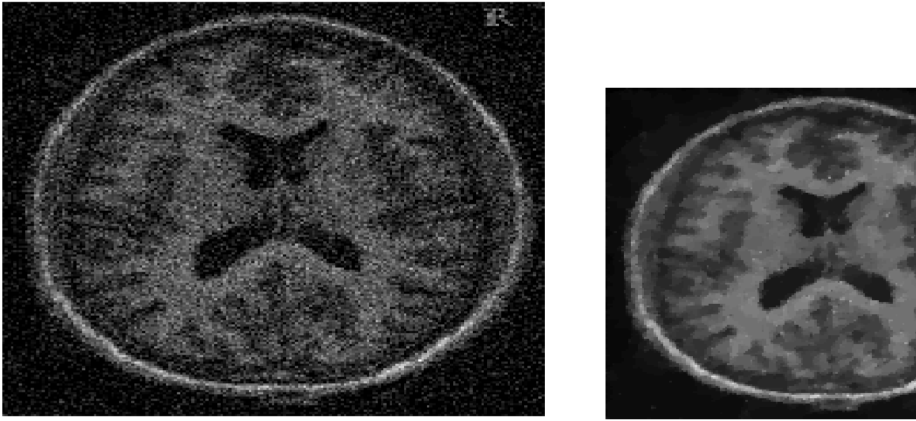

Figure: Noisy and denoised magnetic resonance image of a brain using total variation regularisation.

Image restoration/ denoising . . . ROF Model (Rudin, Osher and Fatemi 1992)

$\Omega \subset \mathbb{R}^2$ open bounded domain, Liptschitz boundary . . . image domain
$u_0 : \Omega \to \mathbb{R}$ . . . (noisy) image
λ . . . tuning parameter

$$\min \left\{ |Du|(\Omega)| + \lambda \int_\Omega |u - u_0|^2 dx : u \in BV(\Omega), u - u_0 \in L^2(\Omega) \right\}$$

removes noise while preserving edges

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extended to higher order and/or vectorial setting (RGB color images)
- Gilles Aubert and Pierre Kornprobst 2006
- Tony Chan, Selim Esedoglu, Frederick Park and Andy Yip 2006

but ...

blurring and stair-case effect

Fidelity term? Regularization term?

Here focus on the fidelity term

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• Yves Meyer 2001 ... images with oscillations often treated as texture or noise $\rightsquigarrow$ the G norm

$$\min\left\{|Du|(\Omega)| + \lambda||u-u_0||_G : u \in BV(\Omega), u-u_0 \in L^2(\Omega)\right\}$$

$$G(\Omega;\mathbb{R}^d) := \{v \in L^2(\Omega;\mathbb{R}^d) : v_i = \text{div}\xi_i, \xi \in L^\infty(\Omega;(\mathbb{R}^2)^d, \xi_i \cdot \nu = 0 \text{ on } \partial\Omega\}$$

$$||v||_G := \inf\{||\xi||_{L^\infty} : v_i = \text{div}\xi_i, \ldots\}$$

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If $\Omega \subset \mathbb{R}^2$ is a domain with Lipschitz boundary

$$G(\Omega;\mathbb{R}^d) = \left\{v \in L^2(\Omega;\mathbb{R}^d) : \int_\Omega v(x)\,dx = 0\right\}$$

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Chromaticity-Brightness, CB

$u_0 : \Omega \to [0, +\infty)^3 \setminus \{0\}$ ... color RGB image

$(u_0)_b := |u_0|$ ... intensity

$(u_0)_c := \frac{u_0}{|u_0|} \ldots \in S^2$ ... chromaticity

$$u_0 = (u_0)_b (u_0)_c$$

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$$u_0 = (u_0)_b (u_0)_c$$

And in general

$$u = (u)_b (u)_c$$

$(u_0)_b \sim$ grey-scale image ...so use **Meyer's G-model**

$(u_0)_c \sim$ colored image ...so adopt a **Kang-March-type model** (Sung Ha Kang and Riccardo March 2007) ...weighted harmonic maps

$$\min \left\{ \int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 dx + \lambda \int_\Omega |u_c - (u_0)_c|^2 dx : u_c \in W^{1,2}(\Omega; S^2) \right\}$$

u_0 extended by zero outside Ω

${u_b}^\sigma := G_\sigma \star (u_0)_b \ldots G_\sigma(x) := \frac{L}{\sigma} e^{-\frac{|x|^2}{4\sigma}}$, $A > 0, \sigma > 0$

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Kang-March

$$\min\left\{\int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 dx + \lambda \int_\Omega |u_c - (u_0)_c|^2 dx : u_c \in W^{1,2}(\Omega; S^2)\right\}$$

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Usually

$$g(t) \sim \frac{1}{1 + \left(\frac{t}{a}\right)^2} \quad \text{or} \quad g(t) \sim e^{-\left(\frac{t}{a}\right)^2},\ a > 0$$

$g \sim 0$ where u_b^σ varies fast $\rightsquigarrow$ sharp transitions of u_c

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Kang-March

$$\min\left\{\int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 dx + \lambda\int_\Omega |u_c - (u_0)_c|^2 dx : u_c \in W^{1,2}(\Omega; S^2)\right\}$$

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• u_b^σ . . . a very smooth version of the brightness component . . . should let $\sigma \to 0$

• $\inf_\Omega g(|\nabla u_b^\sigma|) > 0$ for $\sigma > 0$ . . . hence compactness of minimizing sequences in $W^{1,2}(\Omega; \mathbb{R}^3)$

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Consider

$$\inf_{\substack{u_b \in W^{1,1}(\Omega), u_c \in W^{1,2}(\Omega; S^2), \\ u_b - (u_0)_b \in G(\Omega), u_0 - u_c u_b \in G(\Omega; \mathbb{R}^3)}} \left\{ F_0(u_b u_c) + F_1(u_b) + F_2(u_c) \right\}$$

where

$$F_0(u) := |Du|(\Omega) + \lambda_0 \|u - u_0\|_{G(\Omega;\mathbb{R}^3)}$$
$$u \in BV(\Omega;\mathbb{R}^3), u - u_0 \in G(\Omega;\mathbb{R}^3), \lambda_0 \in \mathbb{R}^+$$

$$F_1(u_b) := |Du_b|(\Omega) + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)}$$
$$u_b \in BV(\Omega), u_b - (u_0)_b \in G(\Omega), \lambda_b \in \mathbb{R}^+$$

$$F_2(u_c) := \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, dx + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx$$
$$u_c \in W^{1,2}(\Omega; S^2), \lambda_c \in \mathbb{R}^+$$

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That is . . .

$$\inf\Big\{\int_\Omega |\nabla(u_c u_b)|\,dx + \int_\Omega |\nabla u_b|\,dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\,dx$$
$$+\lambda_v\|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b\|u_b-(u_0)_b\|_{G(\Omega)} + \lambda_c\int_\Omega |u_c-(u_0)_c|^2\,dx\Big\}$$

where

- $u_b \in W^{1,1}(\Omega)$
- $\boxed{u_c \in W^{1,2}(\Omega; S^2)}$
- $u_b - (u_0)_b \in G(\Omega)$
- $u_0 - u_c u_b \in G(\Omega;\mathbb{R}^3)$

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That is . . .

$$\inf\left\{\int_\Omega |\nabla(u_c u_b)|\,dx + \int_\Omega |\nabla u_b|\,dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\,dx \right.$$
$$\left. + \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2\,dx \right\}$$

where

- $u_b \in W^{1,1}(\Omega)$
- $u_c \in W^{1,2}(\Omega; S^2)$
- $u_b - (u_0)_b \in G(\Omega)$
- $u_0 - u_c u_b \in G(\Omega; \mathbb{R}^3)$

And will assume for some $0 < \alpha \leq \beta$

$$(u_0)_b,\ u_b \in [\alpha, \beta] \quad \text{a.e. in}\Omega$$

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Then

$$\alpha \int_\Omega |\nabla u_c|\, dx \le \int_\Omega |\nabla (u_c u_b)|\, dx + \int_\Omega |\nabla u_b|\, dx$$

and if

$$\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}} \subset \Big\{(u_b,u_c) \in W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,2}(\Omega;S^2) : u_b - (u_0)_b \in G(\Omega), \\ u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3)\Big\}$$

is a infimizing sequence then (up to a subsequence) there exist

- $\bar{u}_b \in BV(\Omega;[\alpha,\beta])$
- $\bar{u}_c \in BV(\Omega;S^2)$

such that

$${u_b}^n \overset{\star}{\rightharpoonup} \bar{u}_b \text{ in } BV(\Omega)\,, \quad {u_c}^n \overset{\star}{\rightharpoonup} \bar{u}_c \text{ in } BV(\Omega;\mathbb{R}^3)$$

$$\bar{u}_b - (u_0)_b \in G(\Omega)\,, \bar{u}_b\bar{u}_c - u_0 \in G(\Omega;\mathbb{R}^3)$$

$$\lim_{n\to+\infty} F^{fid}(u_b^n, u_c^n) = F^{fid}(\bar{u}_b, \bar{u}_c)$$

where the Fidelity Term (sum of the three fidelity terms) is

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$$F^{fid}(u_b, u_c) := \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2\, dx$$

So existence of minimizers ... swlsc of the energy $\rightsquigarrow$ swlsc of the regularizing terms

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So existence of minimizers . . . swlsc of the energy $\rightsquigarrow$ swlsc of the regularizing terms

GOAL: Find an integral representation for

$$\inf\left\{ \liminf_{n\to\infty} \int_\Omega h(u_b^n, u_c^n, \nabla u_b^n, \nabla u_c^n)\,dx : \quad \begin{array}{l} u_b^n \in W^{1,1}(\Omega;[\alpha,\beta]), \\ u_b^n \rightharpoonup u_b \text{ in } W^{1,1}(\Omega), \\ u_c^n \in W^{1,2}(\Omega;S^2), \\ u_c^n \rightharpoonup u_c \text{in } W^{1,1}(\Omega;\mathbb{R}^3) \end{array} \right\}$$

$$h(r,s,\xi,\eta) := |\xi| + g(|\xi|)|\eta|^2 + |s\otimes\xi + r\eta|$$

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In general,

$$(\xi,\eta) \mapsto h(r,s,\xi,\eta) = |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|$$

is not quasiconvex

Moreover, for $(r,s) \in [\alpha,\beta] \times S^2$, h satisfies the non-standard growth conditions

$$\frac{1}{C}(|\xi| + |\eta|) \le h(r,s,\xi,\eta) \le C(1 + |\xi| + |\eta|^2),$$

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$$\frac{1}{C}(|\xi| + |\eta|) \leq h(r, s, \xi, \eta) \leq C(1 + |\xi| + |\eta|^2),$$

which leads us to ... the gap problem !
concerning the unconstrained setting
- I. F., Jan Malý 1997
- I. F., Giovanni Leoni and Stefan Müller 2004
- Giuseppe Mingione and Domenico Mucci 2005

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And more!

Admissible sequences must satisfy

$$u_b^n - (u_0)_b \in G(\Omega), \quad u_b^n u_c^n - u_0 \in G(\Omega;\mathbb{R}^3)$$

or, equivalently,

$$\int_\Omega (u_b^n - (u_0)_b)\,dx = 0, \quad \int_\Omega (u_b^n u_c^n - u_0)\,dx = 0$$

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Challenge: To construct a recovery sequence that simultaneously satisfies the manifold constraint and the average restrictions

So . . . singularly perturb the average constraints

Study the asymptotic behavior as $\varepsilon \to 0^+$ of

$$\inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \left\{F^{reg}(u_b,u_c) + F_\varepsilon^{fid}(u_b,u_c)\right\}$$

where

$$F^{reg}(u_b,u_c) := \int_\Omega |\nabla u_b|\, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\, dx + \int_\Omega |\nabla(u_c u_b)|\, dx$$

standard growth conditions: $\int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\, dx \rightsquigarrow \int_\Omega g(|\nabla u_b|)|\nabla u_c| dx$

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The original fidelity term

$$F^{fid}(u_b, u_c) := \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx$$

$$\begin{aligned} F_\varepsilon^{fid}(u_b, u_c) := \quad & \lambda_v \left\| u_b u_c - u_0 - \fint_\Omega (u_b u_c - u_0) \, dx \right\|_{G(\Omega;\mathbb{R}^3)} \\ & + \frac{1}{\varepsilon} \left| \int_\Omega (u_b u_c - u_0) \, dx \right| \\ & + \lambda_b \left\| u_b - (u_0)_b - \fint_\Omega (u_b - (u_0)_b) \, dx \right\|_{G(\Omega)} \\ & + \frac{1}{\varepsilon} \left| \int_\Omega (u_b - (u_0)_b) \, dx \right| \\ & + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx \end{aligned}$$

Good news:

1. in the limit as $\varepsilon \to 0^+$ we will recover the functional F^{fid}
2. pairs (u_b, u_c) satisfying $u_b - (u_0)_b \in G(\Omega)$ and $u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)$.

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Notation

Recall

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b|\,dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\,dx + \int_\Omega |\nabla(u_c u_b)|\,dx$$
$$= \int_\Omega f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x))\,dx$$

where $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \to [0,+\infty)$

$$f(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta| + |s \otimes \xi + r\eta|$$

$g : [0,+\infty) \to (0,1]$ ... non-increasing, Lipschitz

$g(0) = 1$ and $\lim_{t\to+\infty} g(t) = 0$

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Recession function

$$f^\infty(r, s, \xi, \eta) := \limsup_{t\to+\infty} \frac{f(r, s, t\xi, t\eta)}{t}$$
$$= \limsup_{t\to+\infty} \big(|\xi| + g(t|\xi|)|\eta| + |r\eta + s \otimes \xi|\big)$$
$$= |\xi| + \chi_{\{0\}}(|\xi|)|\eta| + |r\eta + s \otimes \xi|$$

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Tangential Quasiconvex Envelope of f

$T_s(S^2)$...tangential space to S^2 at s

$$\mathcal{Q}_T f(r,s,\xi,\eta) := \inf\left\{ \int_Q f(r,s,\xi+\nabla\varphi(y),\eta+\nabla\psi(y))\,dy : \varphi\in W_0^{1,\infty}(Q),\ \psi\in W_0^{1,\infty}(Q;T_s(S^2))\right\}$$

Recession Function of $\mathcal{Q}_T f$

$$(\mathcal{Q}_T f)^\infty(r,s,\xi,\eta) := \limsup_{t\to+\infty} \frac{Q_T f(r,s,t\xi,t\eta)}{t}$$

Jump Energy Density

$a,b\in[\alpha,\beta]\times S^2$, $\nu\in S^1$, Q_ν ...unit cube in $\mathbb{R}^2$ centered at the origin and with two faces orthogonal to ν

$$\begin{aligned} K(a,b,\nu) &:= \inf\left\{\int_{Q_\nu} f^\infty(\varphi(y),\psi(y),\nabla\varphi(y),\nabla\psi(y))\,dy : (\varphi,\psi)\in\mathcal{P}(a,b,\nu)\right\} \\ &= \inf\left\{\int_{Q_\nu}\big(|\nabla\varphi(y)|+|\nabla(\varphi\psi)(y)|+\chi_{\{0\}}(|\nabla\varphi|)|\nabla\psi|\big)\,dy : (\varphi,\psi)\in\mathcal{P}(a,b,\nu)\right\}\end{aligned}$$

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Relaxation of $F^{reg}(u_b, u_c)$

Recall

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b|\, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\, dx + \int_\Omega |\nabla (u_c u_b)|\, dx$$

extend it to $F : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$

$$F(u_b, u_c) := \begin{cases} F^{reg}(u_b, u_c) & \text{if } (u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2), \\ +\infty & \text{otherwise,} \end{cases}$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$

Looking for the lower semicontinuous envelope of F

$\mathcal{F} : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$

$$\mathcal{F}(u_b, u_c) \quad := \inf \Big\{ \liminf_{n \to +\infty} F(u_b^n, u_c^n) : n \in \mathbb{N},\ (u_b^n, u_c^n) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3), \\ u_b^n \to u_b \text{ in } L^1(\Omega),\ u_c^n \to u_c \text{ in } L^1(\Omega; \mathbb{R}^3) \Big\}$$

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Integral Representation of $F^{reg}(u_b, u_c)$

Theorem

$$\mathcal{F}(u_b, u_c) = \begin{cases} F^{reg,sc^-}(u_b, u_c) & \textit{if } (u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2), \\ +\infty & \textit{otherwise} \end{cases}$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, *where* $F^{reg,sc^-} : BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) \to \mathbb{R}$

$$\begin{aligned} F^{reg,sc^-}(u_b, u_c) &:= \int_\Omega \mathcal{Q}_T f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x))\, dx \\ &+ \int_{S_{(u_b,u_c)}} K\big((u_b, u_c)^+(x), (u_b, u_c)^-(x), \nu_{(u_b,u_c)}(x)\big)\, d\mathcal{H}^1(x) \\ &+ \int_\Omega (\mathcal{Q}_T f)^\infty(u_b(x), u_c(x), C_1(x), C_{2,3}(x))\, |d|D^c(u_b, u_c)|(x) \end{aligned}$$

- C_1 ... *first row of* $C := \frac{dD^c(u_b,u_c)}{d|D^c(u_b,u_c)|}$
- $C_{2,3}$ ... 3×2 *matrix, last two rows of* C

$$X{:=}\{(u_b,u_c)\in BV(\Omega;[\alpha,\beta])\times BV(\Omega;S^2): u_b-(u_0)_b\in G(\Omega), u_bu_c-u_0\in G(\Omega;\mathbb{R}^3)\}$$

btw . . . it is nonempty . . .

Theorem

$\{\varepsilon_n\}_{n\in\mathbb{N}}\to 0^+$, $\{\delta_n\}_{n\in\mathbb{N}}\to 0^+$

$$\min_{(u_b,u_c)\in X}\left(F^{reg,sc^-}(u_b,u_c)+F^{fid}(u_b,u_c)\right)=\lim_{n\to\infty}\inf_{(u_b,u_c)}\left(F^{reg}(u_b,u_c)+F^{fid}_{\varepsilon_n}(u_b,u_c)\right)$$

If $(\bar{u}_b^n,\bar{u}_c^n)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)$ *is a* δ_n*-minimizer of* $F^{reg}+F^{fid}_{\varepsilon_n}$*, i.e.,*

$$F^{reg}(\bar{u}_b^n,\bar{u}_c^n)+F^{fid}_{\varepsilon_n}(\bar{u}_b^n,\bar{u}_c^n)\le \inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)}\left(F^{reg}(u_b,u_c)+F^{fid}_{\varepsilon_n}(u_b,u_c)\right)+\delta_n,$$

then $\{(\bar{u}_b^n,\bar{u}_c^n)\}_{n\in\mathbb{N}}$ *is sequentially, relatively compact with respect to the weak-$\star$ convergence in* $BV(\Omega)\times BV(\Omega;\mathbb{R}^3)$*. If* $(\bar{u}_b,\bar{u}_c)$ *is a cluster point of* $\{(\bar{u}_b^n,\bar{u}_c^n)\}_{n\in\mathbb{N}}$*, then* $(\bar{u}_b,\bar{u}_c)\in X$ *is a minimizer of* $(F^{reg,sc^-}+F^{fid})$ *in* X *and*

$$F^{reg,sc^-}(\bar{u}_b,\bar{u}_c)+F^{fid}(\bar{u}_b,\bar{u}_c)=\lim_{n\to\infty}\left(F^{reg}(\bar{u}_b^n,\bar{u}_c^n)+F^{fid}_{\varepsilon_n}(\bar{u}_b^n,\bar{u}_c^n)\right)$$

What is new . . .

The relaxation result falls within . . . lower semicontinuity and/or integral representations of lower semicontinuous envelopes for

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

$u \in W^{1,p}(\Omega; \mathcal{M})$, $\mathcal{M} \subset \mathbb{R}^d$ is a (sufficiently) smooth, m-dimensional manifold

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E.g., liquid crystals, micromagnetic, magnetostrictive materials,

- Bernard Dacorogna, IF, Jan Malý, Konstantina Trivisa 1999
- Roberto Alicandro, Antonio Esposito and Chiara Leone 2007
- Jean-François Babadjian and Vincent Millot 2010
- Jerry Ericksen 1990
- Domenico Mucci 2009
- Haïm Brézis, Jean-Michel Coron and Elliot Lieb 1986
- and others

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What is new . . .

Key ingredients are
• density of smooth functions in $W^{1,1}(\Omega; \mathcal{M})$
• projection lemma (as in Alicandro, Esposito and Leone 2007, and also Virga 1994)

BUT
as opposed to Alicandro, Esposito and Leone 2007, Babadjian and Millot 2010, Mucci 2009, etc.

• given $(r, s) \in [\alpha, \beta] \times S^2$, $(\xi, \eta) \in \mathbb{R}^2 \times [T_s(S^2)]^2 \mapsto f(r, s, \xi, \eta) \in \mathbb{R}^+$ is NEVER tangential quasiconvex
• our manifold $\mathcal{M} = [\alpha, \beta] \times S^2$ has boundary
• the recession function f^∞ does not satisfy a hypothesis of the type

$$|f(r, s, \xi, \eta) - f^\infty(r, s, \xi, \eta)| \leq C(1 + |(\xi, \eta)|^{1-m})$$

for some $C > 0$ and $m \in (0, 1)$, for a.e. (r, s) and for all (ξ, η)

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The Tangential Quasiconvex Envelope

Inspired by Dacorogna, F., Malý and Trivisa 1999

Lemma

$r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, $\eta \in [T_s(S^2)]^2$

$$\mathcal{Q}_T f(r, s, \xi, \eta) = \mathcal{Q}\tilde{f}(r, s, \xi, \eta)$$

where

$$\mathcal{Q}\tilde{f}(r, s, \xi, \eta) := \inf \left\{ \int_Q \tilde{f}(r, s, \xi + \nabla\varphi(y), \eta + \nabla\psi(y))\, dy : \varphi \in W_0^{1,\infty}(Q),\ \psi \in W_0^{1,\infty}(Q; \mathbb{R}^3) \right\}$$

$$\tilde{f}(r, s, \xi, \eta) := \begin{cases} f(\tilde{r}, \tilde{s}, \xi, P_{\tilde{s}}\eta)\, \phi(|s|) & \text{if } s \in \mathbb{R}^3 \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

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More About $\tilde{f}(r, s, \xi, \eta)$

$$P_s\eta := (\mathbb{I}_{3\times 3} - s \otimes s)\eta$$

projection of $\mathbb{R}^{3\times 2}$ onto $[T_s(S^2)]^2$ (resp., of $\mathbb{R}^3$ onto $T_s(S^2)$)

$$\tilde{r} := \begin{cases} \alpha & \text{if } r \leq \alpha, \\ r & \text{if } \alpha \leq r \leq \beta, \\ \beta & \text{if } r \geq \beta, \end{cases} \qquad \tilde{s} := \frac{s}{|s|},$$

$\phi \in C^\infty(\mathbb{R}; [0, 1])$ . . . cut-off function s. t.

$$\phi(t) = \begin{cases} 1 & \text{if} \quad t \geq 1 \\ 0 & \text{if} \quad t \leq \frac{3}{4} \end{cases}$$

For all $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$

$$\tilde{f}(r, s, \xi, \eta) = f(r, s, \xi, \eta).$$

Remark. There does **NOT** exist $(r, s) \in [\alpha, \beta] \times S^2$ for which

$$(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \mapsto \tilde{f}(r, s, \xi, \eta)$$

is quasiconvex.

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Proof: Road Map

Blow-up method ... but with several road blocks ...

1. Localization of the Energy: $(u,v) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2)$

$$A \in \mathcal{A}(\Omega) \mapsto \mathcal{F}(u,v;A) := \inf \Big\{ \liminf_{n\to+\infty} \int_A f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x))\, dx : \\ n \in \mathbb{N}, (u_n, v_n) \in W^{1,1}(A;[\alpha,\beta]) \times W^{1,1}(A;S^2), \\ u_n \to u \text{ in } L^1(A), v_n \to v \text{ in } L^1(A;\mathbb{R}^3) \Big\}$$

2. Prove that $\mathcal{F}(u,v;\cdot)$ is the restriction of a Radon measure on Ω to $\mathcal{A}(\Omega)$ a. c. wrt $|D(u,v)|$
3. Look at the Radon-Nikodym derivatives, e.g., $(u,v) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2)$. For $\mathcal{L}^2$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u,v;\cdot)}{d\mathcal{L}^2}(x_0) = \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0))$$

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Projection function $\pi_y : \overline{B(0,1)} \setminus \{y\} \to S^2$ (Alicandro, Esposito and Leone 2007)

$$\pi_y(s) := y + \frac{-y\cdot(s-y) + \sqrt{(y\cdot(s-y))^2 + |s-y|^2(1-|y|^2)}}{|s-y|^2}(s-y)$$

projects $s \in \overline{B(0,1)} \setminus \{y\}$ onto S^2 along the direction $s-y$

$$\pi_{y|S^2} = \mathbb{I}_{S^2}, \quad \nabla\pi_y(s)w = w \quad \text{for } s \in S^2,\ w \in T_s(S^2)$$

Lemma

$A \in \mathcal{A}(\Omega)$, $v \in W^{1,1}(A; \overline{B(0,1)}) \cap C^\infty(A;\mathbb{R}^3)$. *There exists* $y \in B\left(0, \frac{1}{2}\right)$ *s. t.* $\pi_y \circ v \in W^{1,1}(A;S^2) \cap C^\infty(A;S^2)$

$$\int_A |\nabla(\pi_y \circ v)|\,dx \le C\int_A |\nabla v|\,dx.$$

and then approximate with same trace on the boundary:

Lemma

$A \in \mathcal{A}_\infty(\Omega)$, $w = (u,v) \in BV(A;[\alpha,\beta]\times S^2)$.
There exists a sequence $\{\bar{w}_n\}_{n\in\mathbb{N}} \subset W^{1,1}(A;[\alpha,\beta]\times S^2) \cap C^\infty(A;\mathbb{R}\times\mathbb{R}^3)$ *s. t.*

1. $\bar{w}_n = w$ *on* ∂A *for all* $n \in \mathbb{N}$
2. $\lim_{n\to\infty} \|\bar{w}_n - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \limsup_{n\to\infty} \int_A |\nabla \bar{w}_n(x)|\,dx \le \widetilde{C}|Dw|(A)$

Upper Bound for $\mathcal{F}$

$(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. Then for $\mathcal{L}^2$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u, v; \cdot)}{d\mathcal{L}^2}(x_0) \leq \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0))$$

Fix $\varepsilon > 0$. Let $\varphi_\epsilon \in W_0^{1,\infty}(Q)$, $\psi_\varepsilon \in W_0^{1,\infty}(Q; T_{v(x_0)}(S^2))$, extended by periodicity to the whole $\mathbb{R}^2$, be such that

$$\begin{aligned} &\mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)) + \varepsilon \geq \\ &\qquad \int_Q f(u(x_0), v(x_0), \nabla u(x_0) + \nabla \varphi_\varepsilon(y), \nabla v(x_0) + \nabla \psi_\varepsilon(y))\, dy \end{aligned}$$

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$\{\varsigma_k\}_{k\in\mathbb{N}}$ ...decreasing sequence of positive real numbers s. t.

$$B(x_0, 2\varsigma_k) \subset \Omega, \quad |Du|(\partial B(x_0,\varsigma_k)) = |Dv|(\partial B(x_0,\varsigma_k)) = 0$$

$\{\rho_n\}_{n\in\mathbb{N}}$ ...standard mollifiers for $\delta = 1/n$

$$u_n(x) := u * \rho_n, \quad v_n := v * \rho_n$$

Use Lemma:
$v_{n,k} := \pi_{y_{n,k}} \circ v_n \in W^{1,1}(B(x_0,\varsigma_k); S^2) \cap C^\infty(\overline{B(x_0,\varsigma_k)}; \mathbb{R}^3)$,
$y_{n,k} \in B(0,1/2)$ s. t.

$$\int_{A^\varepsilon_{n,k}} |\nabla v_{n,k}(x)|\, dx \leq C_\star \int_{A^\varepsilon_{n,k}} |\nabla v_n(x), dx.$$

where

$$A^\varepsilon_{n,k} := \{x \in B(x_0,\varsigma_k) \colon \text{dist}(v_n(x), S^2) > \delta_\varepsilon/2\}$$

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$$A^{\varepsilon}_{n,k} := \{x \in B(x_0, \varsigma_k) \colon \operatorname{dist}(v_n(x), S^2) > \delta_\varepsilon/2\}$$

with $\delta_\varepsilon > 0$ s. t.

$$s_1, s_2 \in B(v(x_0), \delta_\varepsilon) \Rightarrow |\nabla\Pi(s_1) - \nabla\Pi(s_2)| \leq \frac{\rho_\varepsilon}{2b_\varepsilon}.$$

$$b_\varepsilon := 1 + |\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty$$

$$|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2| \leq a_\varepsilon,\ |\xi_1 - \xi_2|, |\eta_1 - \eta_2| \leq \rho_\varepsilon \ \Rightarrow$$
$$|f(u(x_0), v(x_0), \xi_1, \eta_1) - f(u(x_0), v(x_0), \xi_2, \eta_2)| \leq \varepsilon$$

$$a_\varepsilon := \max\{2 + 2|\nabla u(x_0)| + \|\nabla\varphi_\varepsilon\|_\infty, (\|\nabla\Pi\|_\infty + 1)(2 + 2|\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty)\}$$

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cut-off functions

$\zeta_1 \in C_c^\infty(\mathbb{R};[0,1])$, $\|\zeta_1'\|_\infty \le 2/\delta_\varepsilon$

$$\zeta_1(r) = \begin{cases} 1 & r \in \left(-\frac{\delta_\varepsilon}{4}, \frac{\delta_\varepsilon}{4}\right), \\ 0 & r \notin \left(-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2}\right) \end{cases}$$

$\zeta_2 \in C_c^\infty(\mathbb{R}^3;[0,1])$, $\|\nabla\zeta_2\|_\infty \le 2/\delta_\varepsilon$

$$\zeta_2(s) = \begin{cases} 1 & s \in B\left(0, \frac{\delta_\varepsilon}{4}\right), \\ 0 & s \notin B\left(0, \frac{\delta_\varepsilon}{2}\right) \end{cases}$$

$$u^\varepsilon_{n,k}(x) := u_n(x) + \frac{1}{n}\,\zeta_1(u_n(x) - u(x_0))\,\varphi_\varepsilon(nx)$$

$$v^\varepsilon_{n,k}(x) := v_{n,k}(x) + \frac{1}{n}\,\zeta_2(v_{n,k}(x) - v(x_0))\,\psi_\varepsilon(nx)$$

$$\bar{u}^\varepsilon_{n,k}(x) := \Phi_n(u^\varepsilon_{n,k}(x)), \bar{v}^\varepsilon_{n,k}(x) := \begin{cases} v_{n,k}(x) & \text{if } |v_{n,k}(x) - v(x_0)| \ge \dfrac{\delta_\varepsilon}{2}, \\ \Pi(v^\varepsilon_{n,k}(x)) & \text{if } |v_{n,k}(x) - v(x_0)| < \dfrac{\delta_\varepsilon}{2}. \end{cases}$$

$\Phi_n : \mathbb{R} \to \mathbb{R} \ldots$ projection of $\left[\alpha - \|\varphi_\varepsilon\|_\infty / n, \beta + \|\varphi_\varepsilon\|_\infty / n\right]$ onto $[\alpha, \beta]$

$$\Phi_n(r) := \frac{n(\beta - \alpha)r + (\beta + \alpha)\|\varphi_\varepsilon\|_\infty}{n(\beta - \alpha) + 2\|\varphi_\varepsilon\|_\infty}.$$

$$|\nabla \bar{u}^\varepsilon_{n,k}(x)| \leq C_\varepsilon(1 + |\nabla u_n(x) - \nabla u(x_0)|)$$

$$|\nabla \bar{v}^\varepsilon_{n,k}(x)| \leq C_\varepsilon(1 + |\nabla v_{n,k}(x) - \nabla v(x_0)|)$$

$\{\bar{u}^\varepsilon_{n,k}\}_{n\in\mathbb{N}}$ and $\{\bar{v}^\varepsilon_{n,k}\}_{n\in\mathbb{N}}$ are admissible sequences for $\mathcal{F}(u, v; B(x_0; \varsigma_k))$

$$\frac{d\mathcal{F}(u, v; \cdot)}{d\mathcal{L}^2}(x_0) \leq \limsup_{k\to\infty} \limsup_{n\to\infty} \int_{B(x_0,\varsigma_k)} f(u(x_0), v(x_0), \nabla \bar{u}^\varepsilon_{n,k}(x), \nabla \bar{v}^\varepsilon_{n,k}(x))\, dx$$

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…and after a few estimates conclude that

$$\limsup_{k\to\infty}\limsup_{n\to\infty} ⨍_{B(x_0,\varsigma_k)} f\big(u(x_0), v(x_0), \nabla \bar{u}^{\varepsilon}_{n,k}(x), \nabla \bar{v}^{\varepsilon}_{n,k}(x)\big)\, dx$$

$$\leq \limsup_{k\to\infty}\limsup_{n\to\infty} ⨍_{B(x_0,\varsigma_k)} f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla\varphi_\varepsilon(nx), \nabla v(x_0) + \nabla\psi_\varepsilon(nx)\big)\, dx + \varepsilon$$

$$= \int_Q f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla\varphi_\varepsilon(y), \nabla v(x_0) + \nabla\psi_\varepsilon(y)\big)\, dy + \varepsilon$$

$$\leq \mathcal{Q}_T f\big(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)\big) + 2\varepsilon$$

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HAPPY BIRTHDAY LUISA!

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