Some regularity results for integral functionals with variable growth conditions

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- **F.G.-Antonia Passarelli di Napoli** Higher differentiability of minimizers of variational integrals with variable exponents *Math.Z.* 280 (2015), no.3, 873-892
- **F.G.** A $C^{1,\alpha}$ partial regularity result for integral functionals with $p(x)$-growth condition *Adv.Calc.Var* to appear

$\Omega \subset \boldsymbol{R}^n$, $n > 2$, be a bounded open set

$$\mathcal{F} = \int_\Omega F(x, Du(x))\, dx$$

with F strictly convex and C^2 into the gradient variable and satisfying

$$|\xi|^{p(x)} \leq F(x,\xi) \leq c(1 + |\xi|^{p(x)})$$

the so-called $p(x)$-growth condition, where

$$p(x) : \Omega \to (1, +\infty)$$

is a continuous weakly differentiable function such that

$$1 < p_- \leq p(x) \leq p_+ < +\infty$$

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Recall that

$$L^{p(x)}(\Omega, \boldsymbol{R}^N) = \left\{ f : \Omega \to \boldsymbol{R}^N : \int_\Omega |f(x)|^{p(x)} dx < +\infty \right\}$$

and that it is a Banach space equipped with the Luxemburg norm

$$||f||_{L^{p(x)}(\Omega, \boldsymbol{R}^N)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

while, denoting by Df the distributional gradient of f,

$$W^{1,p(x)}(\Omega, \boldsymbol{R}^N) = \{f \in L^{p(x)}(\Omega, \boldsymbol{R}^N) \quad \text{and} \quad Df \in L^{p(x)}(\Omega, \boldsymbol{R}^{N\times n})\}$$

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$$||f||_{W^{1,p(x)}(\Omega,\boldsymbol{R}^N)} := ||f||_{L^{p(x)}(\Omega,\boldsymbol{R}^N)} + ||Df||_{L^{p(x)}(\Omega,\boldsymbol{R}^{N\times n})}$$

Therefore,
if u is a local minimizer of $\mathcal{F}$ and we assume a $p(x)$-growth condition on the integrand F

$$\Downarrow$$

$$|Du(x)| \in L^{p(x)}_{loc}(\Omega)dx$$

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MOTIVATIONS

- Connections with problems emerging from mathematical physic
- The $p(x)$-growth condition is a borderline case between the standard p-growth condition and the non standard (p, q)-growth condition introduced by Marcellini in 1989 and widely studied by many authors

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As **V.V. Zhikov** observed in his earliest paper (*Russ. J. Math.Phys.* (1997)) for the local minimizers of the model functional

$$\int_{\Omega} |Du(x)|^{p(x)} dx,$$

no higher integrability can be expected without assuming $p(x)$ having at least logarithmic modulus of continuity , i.e.

$$|p(x) - p(y)| \leq \omega(|x - y|)$$

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$C^{0,\alpha}$ or even $C^{1,\alpha}$ under suitable regularity assumption for $p(x)$

- **E.Acerbi-G.Mingione** *Archiv.Ration.Mech.Anal.* (2001)
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- **A.Coscia-G.Mingione** *C.R.Math.Acad.Sci.Paris* (1999)
- **M.Eleuteri** *Boll.Unione Mat.Ital. Sez.B* (2004)
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The novelty in our papers is $p(x) \in W^{1,1}_{loc}(\Omega)$ such that

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The idea goes back to some papers where, in the case $p(x) \equiv p$, the dependence of the integrand with respect to the x-variable is assumed to be discontinuous through a suitable Sobolev function, see for example

- **F.Giannetti- A.Passarelli di Napoli** . *Ann. Acad. Sci. Fennicae Math.* (2014)
- **R.Giova** *J. Differential Equations* (2015)
- **J.Kristensen-G.Mingione** . *Arch. Ration. Mech. Anal.* (2010)
- **A.Passarelli di Napoli** *Adv.Calc.Var.* (2014); *NoDEA* (2015)

REMARK on the assumption on $p(\cdot)$: When we deal with the regularity of local minimizers, for $p(x) \equiv p$, we usually read

$$|D_\xi F(x,\xi) - D_\xi F(y,\xi)| \le \omega(|x-y|)(1+|\xi|^2)^{\frac{p-1}{2}}$$

with ω hölder continuous and , in the case of functionals with discontinuous dependence with respect to the x-variable,

$$|D_\xi F(x,\xi) - D_\xi F(y,\xi)| \le (m(x)+m(y))|x-y|(1+|\xi|^2)^{\frac{p-1}{2}}$$

for a non negative function $m \in L^q$, $q \ge 1$

($\Longleftrightarrow D_\xi F(x,\xi) \in W^{1,q}$).
Hence

$$|D_x D_\xi F(x,\xi)| \le L|Dp(x)|(1+|\xi|^2)^{\frac{p(x)-1}{2}} \log(e+|\xi|^2)$$

replaces in a natural way the two conditions above in the case of variable exponent growth condition

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our Orlicz-Sobolev type condition on the exponent function $p(\cdot)$

QUESTION 1: Can we deduce something on the second derivatives of the minimizers under our assumption on $p(\cdot)$, as well as in the case $p(x) \equiv p$?

Actually **A.Canino, P.Le** and **B.Sciunzi** (*Manuscripta Math.*, 2013) proved the higher differentiability of the solutions to a $p(x)$-Laplace equation of the type

$$-\mathrm{div}(|Du|^{p(x)-2}Du) = f$$

but under the stronger assumption $p(\cdot) \in C^1$

THEOREM [F.G.-A.Passarelli di Napoli]

Let $u \in W^{1,1}_{loc}(\Omega, \boldsymbol{R}^N)$ be a local minimizer of $\mathcal{F}$. Assume

- $\xi \to F(x, \xi)$ strictly convex and of class C^2 for a.e. $x \in \Omega$
- F satisfying a $p(x)$-growth condition with $p(x) \in W^{1,1}_{loc}(\Omega)$ such that

$$|Dp(x)| \in L^n \log^{2n-1} L_{loc}(\Omega).$$

- there exists a constant $L > 0$ such that

$$|D_x D_\xi F(x, \xi)| \leq L|Dp(x)|(1 + |\xi|^2)^{\frac{p(x)-1}{2}} \log(e + |\xi|^2) \quad (1)$$

for a.e. $x \in \Omega$ and all $\xi \in \boldsymbol{R}^{N \times n}$

Then

$$(1+|Du|^2)^{\frac{p(x)-2}{4}}|D^2u| \in L^2_{loc}(\Omega).$$

Furthermore, $\exists R_0 = R_0(n, N, L, p_-, p_+)$ such that, whenever $B_R \subset B_{2R} \subset B_{R_0} \Subset \Omega$, the Caccioppoli type inequality

$$\int_{B_R} (1+|Du|^2)^{\frac{p(x)-2}{2}}|D^2u|^2 dx \le \frac{c}{R^2}\int_{B_{2R}} F(x, Du)\, dx$$

holds for a constant $c = c(n, N, L, p_-, p_+)$ and

$$\int_{B_R} |Du|^{\frac{p(x)n}{n-2}} dx \le \frac{c}{R^{\frac{2n}{n-2}}}\left(\int_{B_{2R}} F(x, Du)\, dx\right)^{\frac{n}{n-2}} + \tilde{c}$$

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Sketch of the proof:

Step 1: We establish higher differentiability estimates for the minimizers of approximating functionals constructed adding singular higher order perturbations to the integrand

$$\int_{\Omega'} F(x, Du) + \frac{\varepsilon}{2}|D^k u|^2 dx$$

$\Omega' \Subset \Omega$, $k \in \mathbb{N}$ so large that we have the continuous embedding $W^{k,2}(\Omega', \mathbb{R}^N) \hookrightarrow C^2(\overline{\Omega}', \mathbb{R}^N)$

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Note that these approximating functionals admit a minimizer

$$u_\varepsilon \in W^{k,2}(\Omega', \boldsymbol{R}^N) \cap W^{1,p(x)}_{\tilde{u}_\varepsilon}(\Omega', \boldsymbol{R}^N)$$

$(\tilde{u}_\varepsilon = \Phi_\varepsilon \star u)$ thanks to the direct methods.

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REMARK Actually, the weaker assumption $|Dp(x)| \in L^n \log^n L_{loc}(\Omega)$ could be sufficient to obtain the a priori estimate but we need to assume $|Dp(x)| \in L^n \log^{2n-1} L_{loc}(\Omega)$ in order to approximate $u \in W^{1,p(x)}$ with regular functions. Indeed

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QUESTION 2: Since an higher differentiability result for the local minimizers holds, can we deduce some higher regularity ?

THEOREM [F.G.]:

Let $u \in W^{1,1}_{loc}(\Omega, \boldsymbol{R}^N)$ be a local minimizer of the integral functional

$$\mathcal{F} = \int_\Omega (1 + |Du|^2)^{\frac{p(x)}{2}}\, dx,$$

with $p(x) \in W^{1,1}_{loc}(\Omega)$ such that $p(x) \geq 2$ a.e. in Ω and

$$|Dp(x)| \in L^n \log^{2n-1} L_{loc}(\Omega).$$

Then there exists an open subset Ω_0 of Ω such that

$$\mathrm{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\alpha}_{loc}(\Omega_0, \boldsymbol{R}^N) \quad \forall 0 < \alpha < 1$$

The idea in the proof is to compare the local minimizer of our functional in a ball $B(x, r)$ with a function v whose distributional gradient is regular

Sketch of the proof:

Rescaling the minimizer u on suitable balls we get a sequence v_j weakly converging in $W^{1,2}$ to a function v solution of a linear elliptic system with constant coefficients

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with the aid of the higher differentiability and the higher integrability obtained in [GP], we are able to get a Caccioppoli type inequality for the functions v_j

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Hence

a decay estimate for the excess function

$$E(x_0, r) = \fint_{B_r(x_0)} \left(|Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^{p_+(x_0,r)}\right) dx + r^\beta$$

where $0 < \beta < 2$ and $p_+(x_0, r) = \sup_{B_r(x_0)} p(x)$

From a standard iteration argument , the desired $C^{1,\alpha}$ partial regularity result follows.

PROBLEM 1: The Caccioppoli type inequality depends on the norm of the exponent function $p(\cdot)$

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we make it uniform with respect to the rescaling procedure assuming

$$\mathcal{P}_{x_0,r} := \fint_{B_r(x_0)} |Dp|^n \log^{2n-1}\left(e + \frac{|Dp|}{||Dp||_n}\right) dx \leq K$$

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the singular set of the local minimizers takes into account also this condition:

$$\Omega_0 = \{x \in \Omega : \sup_{R>0} |(Du)_{x_0,R}| < +\infty, \lim_{R\to 0} \mathcal{P}_{x_0,R} < +\infty, \lim_{R\to 0} E(x_0, R) = 0\}$$

PROBLEM 2: Having a growth exponent which is a function in space, the exponent p_+ occurring in the excess function may vary with the ball. It follows that the decay estimate on shrinking balls, involving an exponent that depends on special balls cannot be iterated to achieve the final uniform bound of the excess.

In order to overcome this problem, it was necessary to choose an appropriate large ball B and show the decay estimates on balls $\subset B$ leaving the exponent p_+ unchanged.

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