Asymptotic Development by Gamma Convergence

Calculus of Variations and Its Applications, on the Occasion of Luísa Mascarenhas' 65th Birthday

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December 19, 2015

Papers:

- G. Dal Maso, I. Fonseca and G.L., 2015, Calc. Var. Partial Differential Equations,

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- G. Dal Maso, I. Fonseca and G.L., 2015, Calc. Var. Partial Differential Equations,
- G.L. and R. Murray, 2015, ARMA,
- R. Murray and M. Rinaldi, submitted.

Gamma-Convergence

- De Giorgi (1975), De Giorgi and Franzoni (1975)

Definition

X metric space, $\mathcal{F}_\varepsilon : X \to [-\infty, \infty]$, $\varepsilon > 0$, Γ-converges if there exists $\mathcal{F}^{(0)} : X \to [-\infty, \infty]$ such that

- for every $x \in X$ and every $x_\varepsilon \to x$,

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}^{(0)}(x),$$

We write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}^{(0)}$.

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Gamma-Asymptotic Developments

- Anzellotti and Baldo (1993)

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X metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a Γ-asymptotic development of order k,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

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- $\mathcal{F}_\varepsilon^{(i)} := \dfrac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \overset{\Gamma}{\to} \mathcal{F}^{(i)}$ for $i = 1, \ldots, k$.

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- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subseteq \mathcal{U}_k \subseteq \cdots \subseteq \mathcal{U}_0,$
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Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,

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Let $X = [0,1]$ and

$$\mathcal{F}_\varepsilon(x) = \begin{cases} \varepsilon^n |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

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- Van Der Waals (1893), Cahn and Hilliard (1958), Gurtin (1987)

Zero Order Gamma Limit

Take $X = L^1(\Omega)$, $W^{-1}(\{0\}) = \{\pm 1\}$, and

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) dx$$

if $u \in H^1(\Omega)$, $\int_\Omega u\, dx = m$, and $\mathcal{F}_\varepsilon(u) := \infty$ otherwise in $L^1(\Omega)$.

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- Non-physical solutions.

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- Carr, Gurtin, and Slemrod (1984) for $N = 1$, Modica and Mortola (1979) $W(s) = \sin^2(\pi s)$, Modica (1987), Sternberg (1988) for $N \geq 1$.

First Order Gamma Limit

$$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \int_\Omega u \, dx = m$$

and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

- minimizers of $\mathcal{F}^{(1)}$: $\mathcal{U}_1 = \{1\chi_{E_0} - 1\chi_{\Omega\setminus E_0}\}$, where $E_0 \subset \Omega$ minimizes

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- Gonzalez, Massari and Tamanini (1983), Grüter (1987)

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$$\mathcal{F}_\varepsilon^{(2)} := \frac{\mathcal{F}_\varepsilon^{(1)} - \inf \mathcal{F}^{(1)}}{\varepsilon}$$

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Second Order Gamma Limit, $N = 1$

- Anzellotti and Baldo (1993)
  $W^{-1}(\{0\}) = [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$, where $0 < \delta < 1$,
  $$\int_{-L}^{L} u \, dx = m \rightsquigarrow \quad u(-L) = \alpha, \quad u(L) = \beta.$$

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  $$\frac{\int_{\mathbb{T}} \left(\frac{1}{4\varepsilon}(u_\varepsilon^2 - 1)^2 + \varepsilon |u_\varepsilon'|^2 \right) dx - c_0 \ell}{\varepsilon}$$
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- This inequality is sharp
- Similar result for W of class C^2 quadratic at ± 1.

Second Order Gamma Limit, $N \geq 2$

- Anzellotti, Baldo, and Orlandi (1996) $W(s) = s^2$,

$$\int_\Omega u \, dx = m \rightsquigarrow \quad u = g > 0 \quad \text{on } \partial\Omega.$$

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left(\frac{1}{4\varepsilon}(u^2-1)^2 + \varepsilon|\nabla u|^2\right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}$$

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Theorem (G.L. and Murray)

Let $\Omega \subset \mathbb{R}^N$, $2 \leq N \leq 7$, be open, bounded, of class $C^{2,\alpha}$, $\alpha > 0$. Then for $\mathcal{L}^1$ a.e. mass m,

$$\mathcal{F}^{(2)}(u) = -\frac{(N-1)^2}{9}\kappa^2$$

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- $E_0 \Rightarrow$ Ball compactly contained in Ω.

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Mass–Preserving Allen–Cahn Equation

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u^3 + u + \lambda & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_{0,\varepsilon}(x) & \text{in } \Omega, \end{cases}$$

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$$\int_\Omega \left(\tfrac{1}{4}(u_{0,\varepsilon}^2 - 1)^2 + \varepsilon^2 |\nabla u_{0,\varepsilon}|^2 \right) dx \leq c_0 \operatorname{Per}_\Omega E_0 \varepsilon + C\varepsilon^2,$$

then for every $M > 0$ solutions u_ε of (NAC) satisfy

$$\lim_{\varepsilon \to 0^+} \sup_{0 \leq t \leq M/\varepsilon} \int_\Omega |u_\varepsilon(x,t) - u_0(x)|\, dx = 0.$$

Similar result for Cahn–Hilliard equation.

Slow Motion: Mass–Preserving Allen–Cahn Equation

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Slow Motion

Allen and Cahn (1979)

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- **The Dynamical Approach:** $N = 1$: Fusco and Hale (1989), Carr and Pego (1989), $N \geq 2$: Chen (1992), Kowalczyk (1997), dynamics of a straight interface on a special domain, Ei and Yanagida (1997), dynamics of a straight interface on a strip-like domain, Alikakos, Bronsard, Fusco (1998), dynamics of a ball for nonlocal Allen–Cahn equation

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- **The Energy Approach:** $N = 1$, Bronsard and Kohn (1990), Grant (1995), $N \geq 2$, Bronsard and Kohn (1991), radial case.

Fontanelas

Luisa tomorrow