A new Optimal Transport distance between nonnegative Radon measures

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Static : Monge-Kantorovich problem and Wasserstein distance

$$\mathcal{W}_2^2(\rho_0,\rho_1) = \min_{\gamma\in\Gamma[\rho_0,\rho_1]} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y)$$

Dynamical representation :

Benamou-Brenier formula '00

$$\mathcal{W}_2^2(\rho_0,\rho_1) = \min \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \mathrm{d}\rho_t \mathrm{d}t$$

over curves $\rho\in\mathcal{C}_w([0,1];\mathcal{P}_2)$ and velocity fields $\mathbf{v}\in L^2(0,1;L^2(\mathrm{d}\rho_t))$ s.t.

$$\partial_t\rho_t + \mathrm{div}(\rho_t\mathbf{v}_t) = 0 \qquad \text{(C)}$$

Divergence form $\Rightarrow$ mass conservative, optimal $\mathbf{v}_t = \nabla p_t$

Definition/Theorem (KMV '15)

$\rho_0, \rho_1 \in \mathcal{M}_b^+$:

$$d^2(\rho_0, \rho_1) = \inf \int_0^1 \int_{\mathbb{R}^d} \left(|\nabla u_t|^2 + |u_t|^2\right) \mathrm{d}\rho_t \mathrm{d}t$$

over curves $\rho \in \mathcal{C}_w([0,1]; \mathcal{M}_b^+)$ and potentials $u \in L^2(0,1; H^1(\mathrm{d}\rho_t))$ s.t.

$$\partial_t \rho_t + \mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t \qquad \text{(NCC)}$$

Rmks :

- Reaction term $\rightsquigarrow$ Minimum Principle, mass variations $|\rho_0| \neq |\rho_1|$
- Two strategies : displace or create/kill mass $\Rightarrow$ different geometry than $(\mathcal{P}_2, \mathcal{W}_2)$
- No finite moments or decay required

$$\partial_t \rho_t + \mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t, \qquad d^2(\rho_0,\rho_1) = \inf \int_0^1 \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t$$

Theorem (KMV '15)

(i) $(\mathcal{M}^+, d)$ is a complete metric space

(ii) d is LSC w.r.t weak-$*$ CV of measures and metrizes the narrow topology

(iii) $d \leq \mathcal{W}_2$ on $\mathcal{P}_2$

(iv) compactly supported measures are dense : $d(\rho|_{B_R}, \rho) \to 0$ as $R \to \infty$

Theorem (KMV '15)

(i) Lipschitz curves $d(\rho_t,\rho_s) \leq L|t-s|$ can be represented by potentials $\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L$, and vice-versa.

(ii) $(\mathcal{M}^+, d)$ is a geodesic space, i.e $d^2(\rho_0,\rho_1) = \int_0^1 \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t$ for some constant-speed curve $\|u_t\|_{H^1(\mathrm{d}\rho_t)} = cst$ with $d(\rho_t,\rho_s) = |t-s| d(\rho_0,\rho_1)$.

Rmk : for Wasserstein "narrow $\approx$ weak-$*$". More delicate here, no tightness

Geodesics $\partial_t \rho_t + \mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t$, velocity $\zeta = \frac{d}{dt}\big|_{t=0} \rho_t \sim -\mathrm{div}(\rho \nabla u) + \rho u$.

Differential structure *à la Otto*, $\mathcal{M}_b^+ \approx$ Riemannian manifold

Tangent plane

$$T_\rho \mathcal{M}^+ = \{\zeta = -\mathrm{div}(\rho \nabla u) + \rho u : \quad u \in H^1(\mathrm{d}\rho)\} \qquad (\rho \neq 0 \text{ extremal})$$

and metrics

$$g_\rho(\zeta_1, \zeta_2) = \langle u_1, u_2 \rangle_{H^1(\mathrm{d}\rho)} = \int_{\mathbb{R}^d} (\nabla u_1 \cdot \nabla u_2 + u_1 u_2)\, \mathrm{d}\rho$$

Chain-rule $\frac{d}{dt}\big|_{t=0} \mathcal{F}(\rho_t) = g_\rho(\mathrm{grad}_d \mathcal{F}(\rho), \zeta)$

Gradients

$$\mathrm{grad}_d \mathcal{F}(\rho) = -\mathrm{div}\left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\right) + \rho \frac{\delta \mathcal{F}}{\delta \rho}, \qquad \text{i-e } u = \frac{\delta \mathcal{F}}{\delta \rho}$$

Rmk : 2nd order calculus $g_\rho(\mathrm{Hess}_d \mathcal{F}(\rho) \cdot \zeta, \zeta) = \frac{d^2}{dt^2}\Big|_{geod} \mathcal{F}(\rho_t)$, more involved (generalized sticky particles)

Fitness-driven population dynamics [Cosner, *Th. Pop. Bio.*, '05]

$$\begin{cases} \partial_t \rho = -\operatorname{div}(\rho \nabla (m-\rho)) + \rho(m-\rho), & x \in \Omega \\ \rho \frac{\partial}{\partial \nu}(m-\rho) = 0, & x \in \partial\Omega \\ \rho|_{t=0} = \rho_0(x), & x \in \Omega. \end{cases}$$

Population density $\rho(t,x) \geq 0$, resources $m(x) \geq m_0 > 0$, fitness $m - \rho$.

Ignore BC's, $\operatorname{grad}_d \mathcal{F}(\rho) = -\operatorname{div}\left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\right) + \rho \frac{\delta \mathcal{F}}{\delta \rho}$

$$\boxed{\mathcal{E}(\rho) = \frac{1}{2}\int_\Omega |\rho - m|^2 \mathrm{d}x \qquad \Rightarrow \qquad \frac{d\rho}{dt} = -\operatorname{grad}_d \mathcal{E}(\rho)}$$

Theorem (KMV '15)

There is $\gamma \equiv \gamma(\Omega, m, \rho_0)$ s.t. $\|\rho(t) - m\|_{L^2(\Omega)} \leq e^{-\gamma t}\|\rho_0 - m\|_{L^2(\Omega)}$

Least energy solution $\rho^* = m$, quantitative improvement on [Cosner-Winkler, '14]
No λ-displacement convexity : generalized Beckner inequality, stay away from 0

Thank you for your attention,

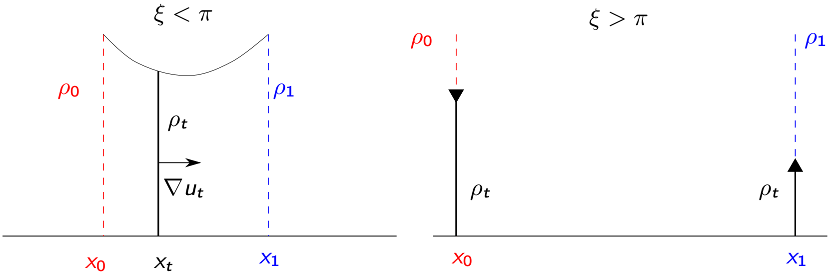
and

Happy birthday Luísa !

An illustrative example

One-point particles $\rho_0 = \delta_{x_0}, \rho_1 = \delta_{x_1}$ at distance $\xi = |x_1 - x_0|$

$$\begin{array}{ll} \xi < \pi : & \text{Wasserstein-like transport } \rho_t = \lambda(t)\delta_{x_t} \\ \xi > \pi : & \text{pure reaction (create/kill) } \rho_t = \kappa(t)\delta_{x_0} + \lambda(t)\delta_{x_1} \\ \xi = \pi : & \text{any combination ! } \rightsquigarrow \text{ non-uniqueness} \end{array}$$



$$\begin{array}{ll} \text{Short range :} & |\mathcal{W}_2(\rho_0,\rho_1) - d(\rho_0,\rho_1)| = o(\xi^2) \\ \text{Long range :} & d(\rho_0,\rho_1) \le d(\rho_0,0) + d(0,\rho_1) \le 4 \end{array}$$

$$\partial_t \rho = -\operatorname{div}(\rho \nabla (m-\rho)) + \rho(m-\rho) \qquad \leftrightarrow \qquad \frac{d\rho}{dt} = -\operatorname{grad}_d \mathcal{E}, \quad \mathcal{E}(\rho) = \frac{1}{2}\int_\Omega |\rho - m|^2$$

1. $\exists!$ and regularity of weak solutions [Cosner-Winkler, '14] : dissipation

$$\mathcal{D}(t) = -\frac{d}{dt}\mathcal{E}(\rho) = |\operatorname{grad}_d \mathcal{E}|^2 = \int_\Omega \left(|\nabla(\rho - m)|^2 + |\rho - m|^2\right) \mathrm{d}\rho$$

2. Entropy/Entropy production : No λ-convexity, generalized Beckner inequality

$$\Phi\left(\int_\Omega f\right)\int_\Omega |f-m|^2 \leq \int_\Omega \left(|\nabla(f-m)|^2 + |f-m|^2\right) f, \qquad \Phi(0) = 0 \text{ and } \Phi \uparrow$$

3. Comparison principle : $\int_\Omega \rho \geq \int_\Omega \min\{\rho_0, m\} = c_0 > 0$, $\gamma = \Phi(c_0)$

$$-\frac{d}{dt}\mathcal{E} = \mathcal{D} \geq 2\lambda\mathcal{E} \qquad \text{and} \qquad \mathcal{E}(t) \leq e^{-2\gamma t}\mathcal{E}(0)$$