Explicit Formulas for Relaxed Disarrangement Densities from Structured Deformations

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CF a SD is a pair (g, G) with $Dg = \nabla g \mathcal{L}^n + [g] \otimes \nu \mathcal{H}^{N-1}$. Approx. Theorem: there exists $f_n \in SBV$ such that $f_n \xrightarrow{L^1} g$, $\nabla f_n \overset{\mathcal{M}}{\rightharpoonup} G$.

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hence the question: does $\mathcal{V}^{|\cdot|}$ have an associated disarrangement density? If so, how does it depend on (g, G)?

Relaxation *à la* Choksi-Fonseca - I

Given an energy density of the form

$$E(u) := \int_\Omega W(\nabla u)\,\mathrm{d}\mathcal{L}^N + \int_{J(u)\cap\Omega} \psi([u],\nu)\,\mathrm{d}\mathcal{H}^{N-1},$$

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The bulk density H and the surface density h are derived via a *cell formula*.

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Theorem (Owen-Paroni, 2015)

Given $\psi : \mathbb{R}^N \times \mathcal{S}^{N-1} \to [0, +\infty)$ *such that*

- *there exists* $C > 0$*:* $\quad 0 \leqslant \psi(\xi, \nu) \leqslant C|\xi|$,
- *for all* $t > 0$, $\quad \psi(t\xi, \nu) = t\psi(\xi, \nu)$,
- *for all* ξ_1, ξ_2, ν, $\quad \psi(\xi_1 + \xi_2, \nu) \leqslant \psi(\xi_1, \nu) + \psi(\xi_2, \nu)$,

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$$\left. u_n \xrightarrow{L^1} g,\ \nabla u_n \overset{*}{\rightharpoonup} G,\ \sup_n (|\nabla u_n|_p + |Du_n|(\Omega)) < +\infty \right\},$$

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we have

$$I(g,G) = \int_\Omega H(\nabla g, G)\,\mathrm{d}\mathcal{L}^N + \int_{J(g)\cap\Omega} h([g],\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

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Theorem (Owen-Paroni, 2015)

The densities H *and* h *are given by*

$$H(A,B) := \inf\left\{ \int_{J(u)\cap Q} \psi([u],\nu)\,\mathrm{d}\mathcal{H}^{N-1} : u \in SBV(Q;\mathbb{R}^N),\right.$$
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where

$$u_{\xi,\eta}(x) := \begin{cases} 0 & \text{if } -1/2 < x\cdot\eta < 0, \\ \xi & \text{if } 0 \leqslant x\cdot\eta < 1/2. \end{cases}$$

Explicit formulas

Theorem (Owen-Paroni (2015))

The choices $\psi^{|\cdot|}(\xi,\nu) := |\xi\cdot\nu|$ *and* $\psi^{\pm}(\xi,\nu) := (\xi\cdot\nu)^{\pm}$ *yield*

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Therefore, the functional for the *minimal volume swept out by disarrangements* reads

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A crucial step in OP's proof was to show that

$$H^{|\cdot|}(A,B) \leqslant |\mathrm{tr}(A-B)|.$$

Our contribution

Given $A, B \in \mathbb{R}^{N\times N}$, there holds

$$|\mathrm{tr}(A-B)| \leqslant \inf\left\{ \int_{J(u)} |[u]\cdot\nu| \, d\mathcal{H}^{N-1} : u \in SBV(Q;\mathbb{R}^N) \right.$$

$$\left. u(x) = Ax \text{ on } \partial Q,\ \nabla u \in L^p(Q),\ \int_Q \nabla u = B \right\}$$

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- Let $\hat{M} := \frac{1}{2}(M+M^\top)$ and choose a o.n. basis $\{e_i\}$: $\hat{M}e_i = \lambda_i e_i$.

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The estimate on the total variation on the jump set of the functions $u_m^{(n)}$ leads to checking if $(\hat{M}Re_i \cdot Re_i) \cdot (\hat{M}Re_j \cdot Re_j) \geqslant 0$ for $i,j = 1,\ldots,N$.

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Thank you for your attention!