A semilinear elliptic problem with a singularity at $u = 0$

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In this joint work with Daniela Giachetti (Rome, Italy) and Pedro J. Martínez Aparicio (Cartagena, Spain) (see [3] and [4]), we consider the semilinear elliptic equation with homogeneous Dirichlet boundary condition

$$-divA(x)Du = F(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad u \geq 0 \quad \text{in } \Omega,$$

where the nonlinearity $F(x,u)$ is singular at $u = 0$, and more precisely where F is a Carathéodory function $F : \Omega \times [0,+\infty[\to [0,+\infty]$ which satisfies

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)} \quad \text{a.e. } x \in \Omega, \forall s > 0,$$

with $h \geq 0$, $h \in L^r(\Omega) \subset H^{-1}(\Omega)$ and $\Gamma : [0,+\infty[\to [0,+\infty[$ a C^1, Lipschitz-continuous, nondecreasing function such that $\Gamma(0) = 0$ and $\Gamma(s) > 0$ for every $s > 0$. A model for such a function $F(x,s)$ is for example given by

$$F(x,s) = \frac{f(x)}{\exp(-\frac{1}{s})}\left(2 + \sin(\frac{1}{s})\right) + \frac{g(x)}{s^\gamma} + l(x) \quad \text{a.e. } x \in \Omega, \forall s > 0,$$

where the functions f, g and l are nonnegative and belong to $L^r(\Omega)$.

The main difficulty is to give a convenient definition of the solution of this problem, in particular when $\Gamma(s) \ll s$ for s close to 0.

We give such a definition and we prove the existence and stability of this solution, as well as its uniqueness when $F(x,s)$ is non increasing in s.

This work has been inspired by the papers [2] of Lucio Boccardo and Luigi Orsina and [1] of Lucio Boccardo and Juan Casado-Diaz.

References:

[1] L. Boccardo & J. Casado-Diaz: Some properties of solutions of some semilinear elliptic singular problems and applications to the G-convergence, *Asymptotic Analysis*, 86 (2104), 1–15.

[2] L. Boccardo & L. Orsina: Semilinear elliptic equations with singular nonlinearities, *Calculus of Variations and Partial Differential Equations*, 37 (2010), 363–380.

[3] D. Giachetti, P.J. Martínez-Aparicio & F. Murat: An elliptic equation with a mild singularity at $u = 0$: existence and homogenization, to appear.

[4] D. Giachetti, P.J. Martínez-Aparicio & F. Murat: Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$, to appear.