On a long range segregation model

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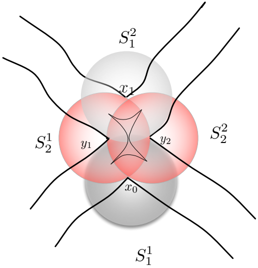
(joint work with Luis Caffarelli and Stefania Patrizi)



Luisa's Conference, 2015

Statement of the problem

$\Omega \subset \mathbb{R}^n$ be a bounded domain where d populations co-exist.

$$\begin{cases} \Delta u_i^\epsilon(x) = \dfrac{1}{\epsilon^2} u_i^\epsilon(x) \displaystyle\sum_{j \neq i} \int_{B_1(x)} u_j^\epsilon(y)\, \mathrm{d}y, \qquad i = 1, \ldots, d, \quad \text{in } \Omega, \\ u_i^\epsilon = \phi_i, \quad i = 1, \ldots, d, \quad \text{on} \quad (\partial\Omega)_1. \end{cases} \tag{1}$$

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where:

- $B_1(x)$ can be replaced by any uniformly convex set and the non-local operator

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- $(\partial\Omega)_1 = \{x \in \Omega^c : \mathrm{dist}(x, \partial\Omega) \leq 1\}$
- $\phi_i \geq 0$ Holder continuous fcts, s.t. $\mathrm{dist}(\mathrm{supp}\, \phi_i, \mathrm{supp}\, \phi_j) \geq 1$, for $i \neq j$

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think about:

- d number of populations that exist in a bounded domain Ω
- u_i^ϵ is the density of the population i, bounded, $0 \leq u_i^\epsilon(x) \leq N$, for all i.
- $\frac{1}{\epsilon^2}$ prescribes the competitive character of the relationship between species

Why is it called a segregation model ?

The simplest model with diffusion (type Gause-Lotka-Volterra):

$$\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + R_1 u_1 - a_1 {u_1}^2 - b_{12} u_1 u_2 \quad \text{in } \Omega,$$

$$\frac{\partial u_2}{\partial t} = \underbrace{d_2 \Delta u_2}_{\text{diffusion term}} + R_2 u_2 - a_2 {u_2}^2 - b_{21} u_1 u_2 \quad \text{in } \Omega,$$

- d_i is the diffusion rate for species i;
- $u_i(x,t)$ is the density of the population i at time t and position x;
- R_i is the intrinsic rate of growth of species i;
- a_i is a positive number that characterizes the intraspecies competition for the species i;
- b_{ij} is a positive number that characterizes the interspecies competition between the species i and j.

Study of existence, uniqueness and regularity for solutions

- P. Korman, A. Lenng '87, A.C. Lazer, P.J. Mckenne '82, C. Gui, Y.Lon '94
- N. Shigesada, K. Kawasaki, E. Terramoto, '84
- M. Minura, S. Ei, Q. Fang '91
- E.N.Dancer '95
- E.N.Dancer, Y. Du '95 '95 '95;

Dancer, Du '95 '95 '95

$$\begin{cases} -\Delta u_i = R_i u_i - a_i {u_i}^2 - \sum_{i \neq j} b_{ij} u_i u_j \quad \text{in } \Omega, \quad i = 1, 2 \\ u_i > 0 \quad \text{in } \Omega, \quad u_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2 \end{cases} \tag{3}$$

Dancer, Hilhorst, Mimura, Peletier '99

associate the spacial segregation obtained when $b_{ij} \to \infty$ with a free boundary problem (two phase FB problem)

Segregation of species with high competition

Pattern formation driven by strong competition ($\epsilon \to 0$)

- E.N.Dancer, D. Hilhorst, M. Minura, L. A. Peletier '99
- M. Conti, S. Terracini, G. Verzini '02 '03 (optimal partition)'05 '06 '08
- M. Conti, V. Felli '06 '08
- E.N.Dancer, Y. Du '03 '06 '08
- L. Caffarelli, F. Lin '08 (variational formulation, optimal partition)
- L. Caffarelli, A.L. Karakhanyan, F. Lin '09 (non variational formulation, viscosity theory)

Fully nonlinear diffusion (Adjacent segregation)

- Fully nonlinear diffusion, V.Q, '13
- Characterization of the free boundary for fully nonlinear diffusion, L. Caffarelli, M. Torres, V.Q, in preparation

Non local segregation models (segregation at distance)

- Nonlocal diffusion, S. Terracini, G. Verzini, A. Zilio, '12 '13
- Linear diffusion and nonlocal interaction, L. Caffarelli, S. Patrizi, V. Q,'14

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Goal:

- Existence and global regularity independent of ϵ
- Study the limit in ϵ and characterize the limit problem

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Goal:

- Existence and global regularity independent of ϵ
- Study the limit in ϵ and characterize the limit problem
  Heuristically, the non local term will force the populations to stay at distance 1, one from each other as ϵ tends to 0.
- Study the regularity of the solution and of the free boundary?

Our results on the long range segregation model

Asymptotic behavior

$$\begin{cases} \Delta u_i^\epsilon(x) = \frac{1}{\epsilon^2} u_i^\epsilon(x) \sum_{j \neq i} \int_{B_1(x)} u_j^\epsilon(y)\,\mathrm{d}y & \Omega \\ u_i^\epsilon = \phi_i & (\partial\Omega)_1 \end{cases} \rightarrow \begin{cases} \Delta u_i = 0, \quad \text{when } u_i > 0 \\ (\operatorname{supp} u_i)_1 \cap \{u_j > 0\} = \emptyset,\ i \neq j \\ u_i \text{ Lipschitz in } \Omega \end{cases}$$

- Existence
- Solutions $(u_i^\epsilon - \epsilon^{\frac{1}{\delta}})^+$ are locally uniformly Lipschitz ind. of ϵ
- Characterization of limit problem

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- Existence
- Solutions $(u_i^\epsilon - \epsilon^{\frac{1}{\delta}})^+$ are locally uniformly Lipschitz ind. of ϵ
- Characterization of limit problem
- Semiconvexity of the free boundary
- The set $\partial\{u_i > 0\}$ has finite $(n-1)$-dimensional Hausdorff measure.
- Sharp characterization of the interfaces
- Classification of the singular sets (n=2)
- Free boundary condition (for B_1)

Semiconvexity of the free boundary

Theorem

If $x_0 \in \partial\{u_i > 0\}$ there is an exterior tangent ball $B_1(y)$ at x_0.
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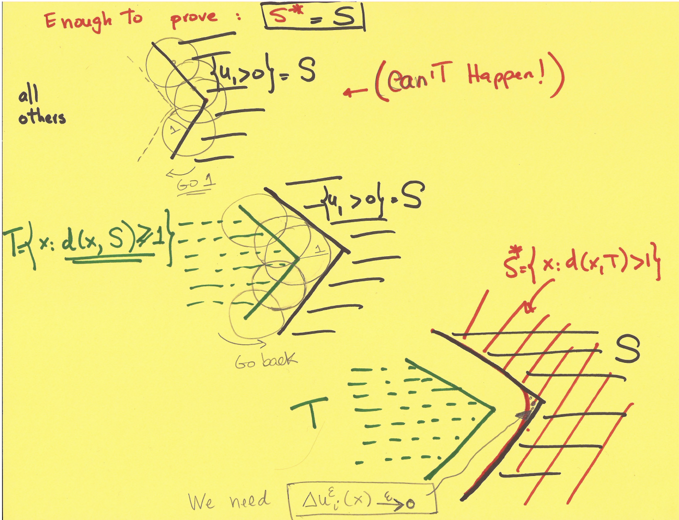
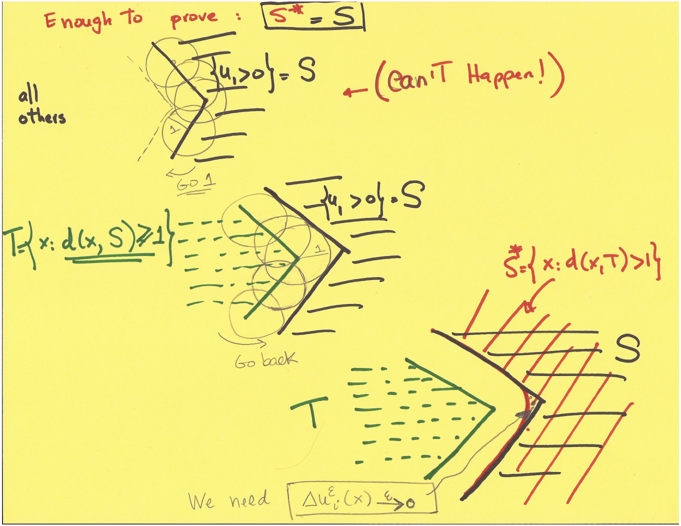


It is enough to prove that $S^*_\sigma \subset S$ and for that it is enough to prove that for all $x \in S^*_\sigma$, $\Delta u_i^\epsilon(x) \to 0$....



Why?

It is enough to prove that for all $x \in S_\sigma^*$, $\Delta u_i^\epsilon(x) \to 0$....

Because:

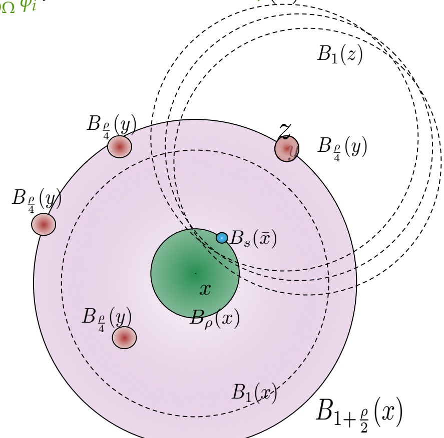
- Since $S_\sigma \subset S_\sigma^*$
- We have $u_i \neq 0$ in S_σ^*
- If $u_i \geq 0$ and u_i is harmonic in S_σ^*, $\Delta u_i = 0$, by strong maximum principle u_i can't have an interior minimum
- So $u_i > 0$ in S_σ^*

To prove $\Delta u_i^\epsilon(x) \to 0$, $x \in S_\sigma^*$, it is enough to prove that:

Claim: If $u_i^\epsilon(x) = m$ then there exists an universal constant τ such that if $y \in B_{1+\frac{\rho}{2}}(x)$, (purple area) then

$$u_j^\epsilon(y) \le ce^{-\frac{c\,m^\alpha r^\beta}{\epsilon}}, \quad j \neq i$$

where $\rho = \frac{\tau m r}{\sup_{\partial\Omega} \phi_i}$, and r such that $B_r(x)$ is far from the boundary.



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Why?

For all $x \in S_\sigma^*$

$$\Delta u_i^\epsilon(x) = \frac{1}{\epsilon^2} u_i^\epsilon(x) \underbrace{\sum_{i \neq j} \int_{B_1(x)} u_j^\epsilon(y) dy}_{\text{same if } x \in S_\sigma}$$

$$\leq C \frac{1}{\epsilon^2} u_i^\epsilon(x) \, |B_1(x)| \, e^{-\frac{c\,m^\alpha r^\beta}{\epsilon}} \to 0$$

Important facts:

Fact 1:

$$\begin{aligned} &\Delta u(x) \geq \theta^2 u(x), \quad x \in B_\rho(0) \\ &u(x) \geq 0, \end{aligned} \quad \Rightarrow \frac{u(0)}{\sup_{B_\rho(0)} u(x)} \leq C\, e^{-c\theta\rho}$$

Fact 2:

$$\begin{aligned} &\Delta u(x) \geq 0, && x \in B_r(0) \\ &u(x) \leq 1, && x \in B_r(0) \\ &u(0) = m > 0 \end{aligned}$$

B ball

$\Rightarrow \exists \tau > 0$ universal const. such that if $\mathrm{dist}(B, 0) \leq \tau m r$ then

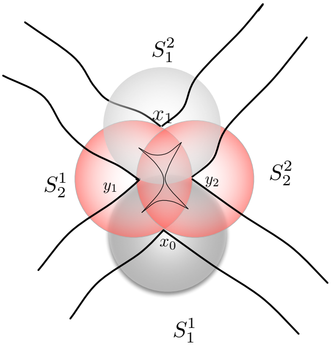
$$\sup_{\partial B \cap B_r(0)} u(x) \geq \frac{m}{2}$$

Classification of the singular sets (n=2)

Theorem

Let $S_i = \{u_i > 0\}$ and consider two points that realize the distance 1 across the boundary, $x_0 \in \partial S_i$ and $y_0 \in \partial S_j$. Assume that they are such that S_i had an angle θ_i at x_0 and S_j has an angle θ_j at y_0. Then

$$\theta_i = \theta_j.$$



Open Problems:

- Regularity for higher dimensions;
- Different weights and shapes of the domain of inter-competition (for instance star-shaped sets);
- Evolution problem associated.

Happy birthday dear Luisa!!
Thank you!