THIN MATERIALS WITH INDUCED METRIC

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- An old story: dimension reduction $3d \to 2d$, many papers in linearized elasticity, nonlinear elasticity with challenging questions,
- More recent: same idea with a different metric.

"A material aims at achieving a prescribed metric."

Is this possible at the 3d-level? What are the induced 2d-models?

Part 1. Classical results, 1993-2006

1.1. 3d-elastic body classical internal elastic energy

$$I(\Phi) = \int_\Omega W(\nabla\Phi(x))\,dx, \quad W : \mathbb{M}^{3+} \mapsto \mathbb{R}^+ \text{ stored energy density,}$$

$$\mathbb{M}^{3+} := \{F \in \mathbb{M}^3; \det F > 0\}.$$

Hypotheses

- Frame indifference $W(F) = \tilde{W}(F^T F)$,
- $W(Id) = 0$; $W(F) = 0 \Leftrightarrow F \in SO(3)$.
  Then, W min on $SO(3)$, $\dfrac{\partial W}{\partial F}(Id) = 0$, Ω natural state.

Under external loads, an equilibrium state in a "minimizer" on $W^{1,p}(\Omega;\mathbb{R}^3)$ of

$$I(\Phi) - \int_\Omega f\cdot\Phi\,dx \quad \text{with b.c.}$$

What are the minimizers when $f = 0$? rigid motions (b.c. compatible)

Indeed,

$$I(\Phi) = \int_\Omega W(\nabla\Phi(x))\,dx, \quad I(\Phi) = 0 \text{ iff } \nabla\Phi(x)^T\nabla\Phi(x) = Id,$$

and Liouville theorem

$$[\Phi : \Omega \mapsto \mathbb{R}^3, \nabla\Phi(x)^T\nabla\Phi(x) = Id, \det\nabla\Phi(x) > 0] \Leftrightarrow [\exists R \in SO(3), \forall x, \Phi(x) = Rx + c].$$

Three proofs

1. $\psi = \Phi - \Phi(0)$, ψ preserves the inner product, hence is linear,
2. differentiate $\partial_i\Phi(x)\cdot\partial_j\Phi(x) = \delta_{ij}$, obtain $\partial_{ij}\Phi(x) = 0$,
3. scheme for more sophisticated results: $\nabla\Phi(x) \in SO(3)$,
   hence $\nabla\Phi = \operatorname{Cof}\nabla\Phi$; $\Delta\Phi = \operatorname{div}\nabla\Phi = \operatorname{div}\operatorname{Cof}\nabla\Phi = 0$ (Piola).
   Thus, Φ is harmonic and C^∞. Now, $|\nabla\Phi(x)|^2 = \operatorname{tr}(\nabla\Phi(x))^T\nabla\Phi(x) = 3$,

   $$0 = \Delta(|\nabla\Phi(x)|^2) = 2|\nabla^2\Phi(x)|^2 + \nabla\Phi(x) : \Delta\nabla\Phi(x) = 2|\nabla^2\Phi(x)|^2.$$

   Order 2 derivatives are null.

1.2. Thin model hierarchy

Find the limit behavior of (almost) minimizers $\Phi^h : \Omega^h = \omega \times]-h, h[\mapsto \mathbb{R}^3$ of

$$\frac{1}{2h} \int_{\Omega^h} W(\nabla \Phi(x))\, dx - \int_{\Omega^h} f^h \cdot \Phi\, dx \quad \text{with b.c.}$$

- hierarchy of four models by asymptotic exp., with no a priori assumption, Fox, R. & Simo, *ARMA, 1993*
  1. nonlinear membrane model
  2. (inextensional) nonlinear bending model
  3. von Kármán model
  4. linear elasticity model

  driven by the loading magnitude, can be expressed in terms of the internal energy magnitude.
- rigorous convergence for the membrane model, Le Dret & R., *JMPA*, 1995
- rigorous convergence for the bending model, Friesecke, James & Müller, *CPAM*, 2002
- rigorous convergence for the vK model, Friesecke, James & Müller, *ARMA*, 2006; R., 2002.

Part 2. Induced metric

2.1. 3d-body with new internal energy, growth-induced

Let be given a

$$\text{metric } G : x \in \Omega \mapsto G(x) \in \mathbb{S}^{3+} = \{\text{positive definite symetric}\}.$$

The material aims at satisfying

$$\nabla\Phi^T \nabla\Phi = G, \ \det(\nabla\Phi(x)) > 0.$$

Modeling proposed by Kupferman, Sharon, *circa* 2008.

Alternative formulation: let $A^2(x) = G(x)$, $A \in \mathbb{S}^{3+}$. There is an internal energy that reads

$$I(\Phi) = \int_\Omega W(\nabla\Phi(x)A^{-1}(x))\, dx, \quad \det(\nabla\Phi(x)) > 0$$

still with $W = 0$ on $SO(3)$. When is this energy 0?

$$\begin{aligned} I(\Phi) = 0 \quad \text{when} \quad & \nabla\Phi(x)A^{-1}(x) \in SO(3), \det(\nabla\Phi(x)) > 0, \\ \textit{i.e.,} \quad & \nabla\Phi^T \nabla\Phi = G, \ \det(\nabla\Phi(x)) > 0. \end{aligned}$$

Is $\nabla\Phi^T\nabla\Phi = G$, $\det(\nabla\Phi(x)) > 0$ possible?

"Easy" : use $\partial_{ik}\partial_j\Phi = \partial_{ij}\partial_k\Phi$ in $\nabla\Phi^T\nabla\Phi = G$. Obtain $\mathscr{R} = 0$ where

$\mathscr{R}_{qijk} = \partial_j\Gamma_{ikq} - \partial_k\Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$, six "independent" nonzero entries,

$$2\Gamma_{ijq} = \partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}, \Gamma^p_{ij} = g^{pq}\Gamma_{ijq}, (g^{pq}) = G^{-1}.$$

Conversely,
fundamental theorem of Riemannian geometry:
if $\mathscr{R} = 0$, there exists Φ, $\det\nabla\Phi(x) > 0$ and $\nabla\Phi^T\nabla\Phi = G$, G said flat metric.

Additional result, Lewicka & Pakzad, 2011

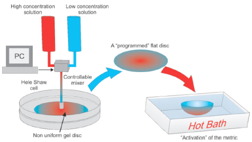
if $\mathscr{R} \neq 0$, we know that $\nexists\Phi \in H^1, ||\operatorname{dist}(\nabla\Phi(\cdot), SO(3)A(\cdot))||_{L^2(\Omega)} = 0$.
In fact,

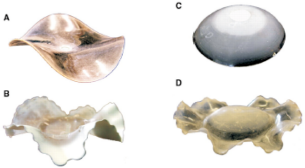
$$\inf_{\Phi\in H^1} ||\operatorname{dist}(\nabla\Phi(\cdot), SO(3)A(\cdot))||_{L^2(\Omega)} > 0.$$

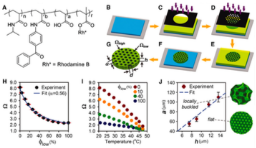
Not obvious, because there is no reason why the inf should be attained.

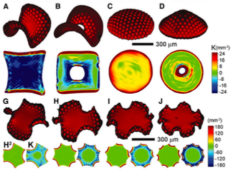
Method of proof: reminiscent of the quantitative rigidity estimate.

2.2. Induced thin models, of interest even in the absence of loads or b.c.









Problem setting: behavior of (almost) minimizers Φ^h of

$$I^h(\Phi) = \frac{1}{2h}\int_{\Omega^h} W(\nabla\Phi(x)A^{-1}(\bar{x}))\,dx,\ \ \Phi : \Omega^h = \omega\times]-h,h[\mapsto \mathbb{R}^3,$$

A only depends on $\bar{x} = (x_1, x_2)$. Magnitude of $\inf I^h$? $I^h(\Phi)$ reads as well

$$I^h(\Phi) = \int_{\Omega} W(\nabla_h\Phi(x)A^{-1}(\bar{x}))\,dx, \quad \Phi : \Omega = \omega\times]-1,1[\mapsto \mathbb{R}^3,$$

where we keep the same notation Φ

$$\nabla_h\Phi = (\partial_1\Phi, \partial_2\Phi, \frac{1}{h}\partial_3\Phi).$$

$$I^h(\Phi) = \int_\Omega W(\nabla_h \Phi(x) A^{-1}(\bar{x}))\, dx, \quad \nabla_h \Phi = (\partial_1 \Phi, \partial_2 \Phi, \frac{1}{h}\partial_3 \Phi)$$

Order 0 model: Generalized membrane energy

Usual case $A = Id$. For $\bar{F} \in \mathbb{M}_{3\times 2}$, let $W_0(\bar{F}) = \min\{W(F); F \in \mathbb{M}^3, F_{3\times 2} = \bar{F}\}$.

$$I^h \xrightarrow{\Gamma - L^p(\Omega)} I_0,\ I_0(\Phi) = \begin{cases} \int_\omega QW_0(\bar{\nabla}\varphi(\bar{x}))\, d\bar{x},\ \Phi = \varphi \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty,\ \Phi \in L^p(\Omega;\mathbb{R}^3) \setminus W^{1,p}(\omega;\mathbb{R}^3). \end{cases}$$

Easily,

$QW_0(\bar{\nabla}\varphi(\bar{x}))$ only depends on $(\nabla\varphi(\bar{x}))^T \nabla\varphi(\bar{x})$ metric tensor

$I_0(\varphi) = 0$ for $\varphi : \omega \mapsto \mathbb{R}^3$ Eucl. isometry $\partial_\alpha \varphi \cdot \partial_\beta \varphi = \delta_{\alpha\beta}$ (and more).

Note that zero-energy deformations for the 3d-model are $SO(3)$, 2d-limit has much more.

Fox, R. & Simo, Le Dret & R., Ben Belgacem.

General $A(\bar{x})$. $\bar{F} \in \mathbb{M}_{3\times 2}$, $W_0(\bar{x}, \bar{F}) := \min\{W\left(FA^{-1}(\bar{x})\right); F \in \mathbb{M}^3, F_{3\times 2} = \bar{F}\}$.

$$I^h \xrightarrow{\Gamma - L^p(\Omega)} I_0,\ I_0(\Phi) = \begin{cases} \int_\omega QW_0(\bar{x}, \bar{\nabla}\varphi(\bar{x}))\, d\bar{x},\ \Phi = \varphi \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty,\ \Phi \in L^p(\Omega;\mathbb{R}^3) \setminus W^{1,p}(\omega;\mathbb{R}^3). \end{cases}$$

Now, $I_0(\varphi) = 0$ as soon as

$$\exists b(\bar{x}) \in \mathbb{R}^3 \text{ s.t} \left(\partial_1\varphi(\bar{x}) | \partial_2\varphi(\bar{x}) | b(\bar{x})\right) A^{-1}(\bar{x}) \in SO(3)$$

i.e, omitting $\bar{x}$,

$$\begin{pmatrix} \bar{\nabla}\varphi^T \bar{\nabla}\varphi & \partial_\alpha \varphi \cdot b \\ \partial_\alpha \varphi \cdot b & |b|^2 \end{pmatrix} = \begin{pmatrix} G_{\alpha\beta} & G_{\alpha 3} \\ G_{\alpha 3} & G_{33} \end{pmatrix}, \ \det(\partial_1 \varphi | \partial_2 \varphi | b) > 0.$$

It suffices to satisfy

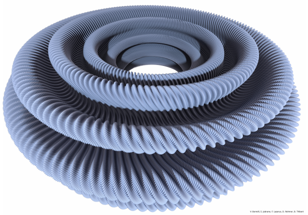
$$\bar{\nabla}\varphi^T \bar{\nabla}\varphi = [G_{\alpha\beta}]_{\alpha,\beta=1,2} := G_{2\times 2}. \qquad (1)$$

Indeed, b follows: given components along two vectors, norm and orientation.

Is (1) realizable? Yes, Nash-Kuiper *circa* 1954, with C^1-regularity, not C^2 (curvature obstruction, Hilbert counter-example, 1901).

Even if the 3d-model has no zero-energy deformations (and $\inf > 0$), 2d-limit has.

Footnote: Isometric immersion of the flat torus into $\mathbb{R}^3$, based on Gromov construction (plenty of cheap solutions), Hevea project.



Order 2 model: Generalized bending energy

Usual case $A = Id$.

For $F^\sharp \in \mathbb{M}_{2\times 2}$, let $W_2(F^\sharp) = \min\{W''(Id)(F,F); F \in \mathbb{M}^3, F_{2\times 2} = F^\sharp\}$,

$$= 2\mu\left|\frac{F^\sharp + F^{\sharp T}}{2}\right|^2 + \frac{2\mu}{2\mu+\lambda}(\mathrm{tr}\, F^\sharp)^2.$$

$$\frac{I^h}{h^2} \xrightarrow{\Gamma - H^1(\Omega)} I_2,\ I_2(\Phi) = \begin{cases} \frac{1}{3!}\int_\omega W_2\left((\bar\nabla\varphi^T\bar\nabla n)(\bar x)\right) d\bar x,\ \Phi = \varphi \in H^2(\omega;\mathbb{R}^3), iso, \\ +\infty \text{ otherwise.}\end{cases}$$

iso: $|\partial_1\varphi| = 1, |\partial_2\varphi| = 1, \partial_1\varphi\cdot\partial_2\varphi = 0$, $\bar\nabla\varphi^T\bar\nabla n$: surface curvature tensor

Obviously,

$I_2(\varphi) = 0$ for $\varphi : \omega \mapsto \mathbb{R}^3$ Eucl. isometry and null curvature tensor (first and second forms equal to 0): $\varphi \in O(3)(\bar x, 0)$.

Fox, R. & Simo, Friesecke, James & Müller, Pantz

Extended $A(\bar x) = \begin{pmatrix} A_{\alpha\beta}(\bar x) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$F^\sharp \in \mathbb{M}_{2\times 2}$, $W_2(\bar x, F^\sharp) = \min\{W''(Id)(A^{-1}FA^{-1})^{(2)}; F \in \mathbb{M}^3, F_{2\times 2} = F^\sharp\}$.

$$\frac{I^h}{h^2} \xrightarrow{\Gamma - H^1(\Omega)} I_2,\ I_2(\Phi) = \begin{cases} \frac{1}{3!}\int_\omega W_2\left(\bar x, ((\bar\nabla\varphi)^T\bar\nabla n)(\bar x)\right) d\bar x,\ \Phi = \varphi \in H^2(\omega;\mathbb{R}^3), iso, \\ +\infty \text{ otherwise.}\end{cases}$$

iso: $(\bar\nabla\varphi)^T\bar\nabla\varphi = G_{2\times 2}$

Lewicka& Pakzad

General $A(\bar{x})$.

$$W_2(\bar{x}, F^\sharp) = \min\{W''(Id)(A^{-1}(\bar{x})\, F\, A^{-1}(\bar{x}))^{(2)}; F \in \mathbb{M}^3, F_{2\times 2} = F^\sharp\}.$$

$$\frac{I^h}{h^2} \xrightarrow{\Gamma - H^1(\Omega)} I_2,\ I_2(\Phi) = \begin{cases} \frac{1}{3!}\int_\omega W_2\left(\bar{x}, ((\bar{\nabla}\varphi^T \bar{\nabla} b)(\bar{x})\right) d\bar{x},\ \Phi = \varphi \in H^2(\omega; \mathbb{R}^3), iso, \\ +\infty \text{ otherwise.} \end{cases}$$

iso: $(\bar{\nabla}\varphi)^T\bar{\nabla}\varphi = G_{2\times 2}$ where $b \in H^1 \cap L^\infty$ uniquely defined in terms of φ as in slides 8 and 9 *i.e.*

$$\begin{pmatrix} \bar{\nabla}\varphi^T\bar{\nabla}\varphi & \partial_\alpha\varphi \cdot b \\ \partial_\alpha\varphi \cdot b & |b|^2 \end{pmatrix} = \begin{pmatrix} G_{\alpha\beta} & G_{\alpha 3} \\ G_{\alpha 3} & G_{33} \end{pmatrix},\ \det(\partial_1\varphi|\partial_2\varphi|b) > 0.$$

Or, letting $Q = \left(\partial_1\varphi(\bar{x})|\partial_1\varphi(\bar{x})|b(\bar{x})\right)$, there holds $Q^T Q = G, \det Q > 0$.

Bhattacharya, Lewicka & Schäffner

Method of proof: extension of the quantitative rigidity estimate on slender domains, F.J.M, 2002

$\Omega \in \mathbb{R}^3$ given: $\exists C(\Omega) > 0$,

$$\forall \Phi \in H^1(\Omega; \mathbb{R}^3), \underset{\text{indep. } x}{\exists R \in SO(3)}, ||\nabla\Phi - R||_{L^2(\Omega)} \le C(\Omega)||\text{dist}(\nabla\Phi, SO(3))||_{L^2(\Omega)}$$

$\Omega^h = \omega \times]-h, h[$ or alternatively, ∇_h on Ω: roughly speaking, $\exists c(\omega) > 0$,

$$\forall h, \forall \Phi \in H^1(\Omega; \mathbb{R}^3), \exists R : \omega \mapsto SO(3), \begin{cases} ||\nabla_h\Phi - R||_{L^2(\Omega)} \le c||\text{dist}(\nabla_h\Phi, SO(3))||_{L^2(\Omega)} \\ ||\nabla R||_{L^2(\omega)} \le \frac{c}{h}||\text{dist}(\nabla_h\Phi, SO(3))||_{L^2(\Omega)} \end{cases}$$

(bis)

$$\frac{I^h}{h^2} \xrightarrow{\Gamma - H^1(\Omega)} I_2,\ I_2(\Phi) = \begin{cases} \frac{1}{3!}\int_\omega W_2\left(\bar{x}, (\bar{\nabla}\varphi^T \bar{\nabla} b)(\bar{x})\right) d\bar{x},\ \Phi = \varphi \in H^2(\omega;\mathbb{R}^3), iso, \\ +\infty \text{ otherwise.} \end{cases}$$

If $\min I_2 = 0$, further information may be sought for.

$$\min I_2 = 0 \Leftrightarrow \exists \varphi \in H^2(\omega;\mathbb{R}^3),\ \bar{\nabla}\varphi(\bar{x})^T \bar{\nabla}\varphi(\bar{x}) = G_{2\times 2},\ \bar{\nabla}\varphi^T \bar{\nabla} b \text{ skew}$$

$$\Leftrightarrow \mathscr{R}_{1212} = \mathscr{R}_{1213} = \mathscr{R}_{1223} = 0\ \ (\neq [\mathscr{R} = 0], \inf I^h > 0 \text{ allowed}).$$

Such φ is unique, because its 2nd fundamental form, in addition to its first fundamental form, is then given in terms of G.

Order 4 model: Generalized von Kármán enegy

Usual case $A = Id$. Recall that

$$\frac{I^h}{h^2} \xrightarrow{\Gamma - H^1(\Omega)} I_2,\ I_2(\varphi) = [\text{second fundamental form}]^2 \text{ on } \mathrm{iso}(\omega; \mathbb{R}^3),$$

$$= 0 \text{ for } \varphi = R(\bar{x}, 0),\ R \in O(3)$$

Examine smaller orders of magnitude of I^h corresponding intuitively to

$$\text{“}\Phi^h(x) = (\bar{x}, 0) + h(u_1^1, u_2^1, x_3 + u_3^1) + h^2 \Phi^2 + \cdots .\text{”}$$

Obtain $u_1^1 = u_2^1 = 0$ (through $\partial_\alpha u_\beta^1 + \partial_\beta u_\alpha^1 = 0$), $\Phi^2 = u^2 - x_3 \partial_\alpha u_3^1 e_\alpha$,

$$\frac{I^h}{h^4} \to I_4,\ I_4(u_1^2, u_2^2, u_3^1) = \int_\omega \left(2W_2 \left(\frac{\partial_\alpha u_\beta^2 + \partial_\beta u_\alpha^2 + \partial_\alpha u_3^1 \partial_\beta u_3^1}{2} \right) + \frac{1}{3!} W_2(\partial_{\alpha\beta} u_3^1) \right) d\bar{x},$$

recall that W_2 is quadratic in its argument.

Fox, R. & Simo, Friesecke, James & Müller, R.

General $A(\bar{x})$. Lewicka, Raoult & Ricciotti. Start from $\min I_2 = 0$,

i.e. $\mathscr{R}_{1212} = \mathscr{R}_{1213} = \mathscr{R}_{1223} = 0$,

i.e. $\exists \varphi \in H^2(\omega;\mathbb{R}^3)$, $\bar{\nabla}\varphi^T\bar{\nabla}\varphi = G_{2\times 2}$ and $\bar{\nabla}\varphi^T\bar{\nabla} b$ skew,

where $b : \omega \mapsto \mathbb{R}^3$ defined by

$Q = \left(\partial_1\varphi(\bar{x})|\partial_1\varphi(\bar{x})|b(\bar{x})\right), Q^TQ = G, \det Q > 0$, or else $b = -(G^{33})^{-1}(G^{\alpha 3}\partial_\alpha\varphi) + (G^{33})^{-1/2} n$.

First finding. Then $\inf I^h$ is indeed smaller: $\inf I^h \leq Ch^4$.

Let simply $\Phi^h(\bar{x}, x_3) = \varphi(\bar{x}) + x_3 b(\bar{x}) + \frac{x_3^2}{2} d(\bar{x})$ (indep. of h). Which d?

$$\nabla\Phi^h(\bar{x}, x_3) = Q(\bar{x}) + x_3 B(\bar{x}) + \frac{x_3^2}{2} D(\bar{x}),$$

with

$$Q = [\partial_\alpha\varphi, b],\ B = [\partial_\alpha b, d],\ D = [\partial_\alpha d, 0].$$

$$\begin{aligned}
\nabla\Phi^h A^{-1}(\bar{x}, x_3) &= QA^{-1}(\bar{x}) + x_3 BA^{-1}(\bar{x}) + \frac{x_3^2}{2} DA^{-1}(\bar{x}) \\
&= (QA^{-1})(\mathrm{Id} + x_3 A^{-1}Q^T BA^{-1} + x_3^2 T). \\
W(\nabla\Phi^h A^{-1}) &= W(\mathrm{Id} + x_3 A^{-1}Q^T BA^{-1} + x_3^2 T).
\end{aligned}$$

Make $Q^T B$ skew-symmetric (will kill the x_3^2 term in $W''(Id)$).

$$Q=[\partial_\alpha\varphi,b],\ B=[\partial_\alpha b,d],\ Q^TB=\begin{pmatrix}\bar\nabla\varphi^T\bar\nabla b & \bar\nabla\varphi^T d\\ b^T\bar\nabla b & b\cdot d\end{pmatrix},$$

b is already determined, we know that $\bar\nabla\varphi^T\bar\nabla b$ is skew,
then choose d s.t. Q^TB skew: $Q^Td=(-b\cdot\partial_1 b,-b\cdot\partial_2 b,0)^T$.

Limit model. We already know that $\Phi^h \xrightarrow{H^1} \varphi$, $\frac{1}{h}\partial_3\Phi^h \xrightarrow{L^2} b$. Now,

$$U^h(\bar{x}) := \frac{1}{2h}\int_{-1}^{1}\Big(\Phi^h - \big(\varphi + hx_3 b\big)\Big)\,dx_3 \xrightarrow{H^1} u^1,\ \mathrm{sym}\left(\bar\nabla\varphi^T\bar\nabla u^1\right) = 0,$$

analog of $\partial_\alpha u^1_\beta + \partial_\beta u^1_\alpha = 0$

$$\frac{1}{h}\,\mathrm{sym}\left(\bar\nabla\varphi^T\bar\nabla U^h\right) \to e^2 \in L^2(\omega;\mathbb{S}_2),$$

analog of $\frac{\partial_\alpha u^2_\beta + \partial_\beta u^2_\alpha}{2}$

Limit energy given by

$$\frac{\partial_\alpha u^2_\beta + \partial_\beta u^2_\alpha + \partial_\alpha u^1_3\,\partial_\beta u^1_3}{2}$$

$$\begin{aligned} I_4(u^1,e^2) = I_{vK}(u^1,e^2) &= 2\int_\omega W_2\Big(\bar{x}, e^2 + \frac{1}{2}(\bar\nabla u^1)^T\bar\nabla u^1 + \frac{1}{4!}\bar\nabla b^T\bar\nabla b\Big) \\ &+ \frac{1}{3!}\int_\omega W_2\Big(\bar{x}, \overbrace{\bar\nabla\varphi^T\bar\nabla p^1}^{-\partial_{\alpha\beta}u^1_3} + (\bar\nabla u^1)^T\bar\nabla b\Big) \\ &+ \frac{2}{6!}\int_\omega W_2\Big(\bar{x}, \mathrm{sym}(\bar\nabla\varphi^T\bar\nabla d) + \bar\nabla b^T\bar\nabla b\Big) \end{aligned}$$

where p^1 defined by $Q^T p^1 = (-b\cdot\partial_1 u^1, -b\cdot\partial_2 u^1, 0)^T$. usual case $p^1 = (-\partial_\alpha u^1_3, 0)$

Remark: the last term is constant.

Proof: Not so simple.

- We have to study sequences u^h such that $I^h(u^h) \leq Ch^4$ and prove some compactness. We first prove that their gradients are locally close to $Q(\bar{x}) + x_3 B(\bar{x})$

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x) - R^h(\bar{x})(Q(\bar{x}) + x_3 B(\bar{x})|^2 \ dx \leq Ch^4 \tag{2}$$

and the variation of $R^h(\bar{x})$ is controlled

$$\int_{\omega} |\nabla R^h(\bar{x})|^2 \, d\bar{x} \leq Ch^2. \tag{3}$$

- Then, we prove existence of constants rotations R^h and vectors c^h such that $y^h = R^h u^h - c^h$ enjoys the above properties.

(bis) Limit energy given by

$$
\begin{aligned}
I_{vK}(u^1, e^2) \quad &= \quad 2\int_\omega W_2\Big(\bar{x}, e^2 + \overset{\frac{\partial_\alpha u^2_\beta + \partial_\beta u^2_\alpha + \partial_\alpha u^1_3 \partial_\beta u^1_3}{2}}{\frac{1}{2}(\bar\nabla u^1)^T \bar\nabla u^1} + \frac{1}{4!}\bar\nabla b^T \bar\nabla b\Big) \\
&+ \quad \frac{1}{3!}\int_\omega W_2\Big(\bar{x}, \overset{-\partial_{\alpha\beta} u^1_3}{\bar\nabla \varphi^T \bar\nabla p^1} + (\bar\nabla u^1)^T \bar\nabla b\Big) \\
&+ \quad \frac{2}{6!}\int_\omega W_2\Big(\bar{x}, \mathrm{sym}(\bar\nabla \varphi^T \bar\nabla d) + \bar\nabla b^T \bar\nabla b\Big)
\end{aligned}
$$

where $p^1 = p^1(u^3)$ (usual case $p^1 = (-\partial_\alpha u^1_3, 0)$.

The third term is constant determined by the previous steps and

$$
\mathrm{sym}(\bar\nabla \varphi^T \bar\nabla d + \bar\nabla b^T \bar\nabla b) = \begin{pmatrix} \mathscr{R}_{1313} & \mathscr{R}_{1323} \\ \mathscr{R}_{1323} & \mathscr{R}_{2323} \end{pmatrix}.
$$

Therefore, the third term is 0 iff $\mathscr{R} = 0$, *i.e* the 3d metric is flat. All minima including those of the 3d-problem are 0.

An example: $G(x', x_3) = \mathrm{diag}(1, 1, \lambda(x'))$

(i) G is immersible in $\mathbb{R}^3$ if and only if

$$M_\lambda = \nabla^2\lambda - \frac{1}{2\lambda}\nabla\lambda \otimes \nabla\lambda \equiv 0 \quad \text{in } \omega,$$

(ii) The Γ-limit energy functional I_{vK} becomes

$$\forall w \in W^{1,2}(\omega, \mathbb{R}^2), \quad \forall v \in W^{2,2}(\omega, \mathbb{R}),$$

$$I_{vK}\, v, w) = 2\int_\Omega W_2\big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v + \frac{1}{96\lambda}\nabla\lambda \otimes \nabla\lambda\big)\ \mathrm{d}x'$$
$$+ \frac{1}{3!}\int_\Omega W_2\big(\sqrt{\lambda}\nabla^2 v\big) + \frac{1}{2 \times 6!}\int_\Omega W_2\big(M_\lambda\big)\ \mathrm{d}x',$$

where W_2 is independent of x'.