Critical speed to travelling waves in models with nonlinear diffusion

Luís Sanchez

CMAF-CIO and Dep. Mathematics, FCUL

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- A very short and arbitrary list of references:
  - A. Kolmogorov, I. Petrovski, and N. Piscounov, Bull. Univ. Moskou Ser. Internat. Sec. A (1937)
  - D. G. Aronson and H. F. Weinberger, Adv. in Math. **30** (1978)
  - Mark J. Ablowitz, A. Zeppetella, Bull. Math. Biol. 41 (1979)
  - L. Malaguti, C. Marcelli, Math. Nachr., **242** (2002)
  - L. Malaguti, C. Marcelli, Matucci, Abstract and Applied Analysis (2011)
  - Murray, Mathematical Biology 2002.
  - M. Arias, J. Campos, A. Robles Pérez, L. Sanchez, CVPDE, (2004)
  - D. Bonheure, L. Sanchez, Handbook of Differential Equations: Ordinary Differential Equations, vol. 3, (2006).

This talk is about results from joint work with

R. Enguiça and A. Gavioli (DCDS 2013),

I. Coelho (AMC 2014),

M. Garrione (BVP 2015) and

A. Gavioli (AML 2015).

There are very recent results of Alessandro Audrito and Juan Luis Vásquez sharing some common aspects with parts of this presentation.

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$$y' = q(h(c,u)\,y^{\frac{1}{p}} - f(u)), \quad 0 \leq u \leq 1 \tag{1}$$

- where c is a parameter, $p > 1$, q is the conjugate of p:

$$\frac{1}{p} + \frac{1}{q} = 1,$$

  $h \colon \mathbb{R} \times [0, 1] \to \mathbb{R}$ is a continuous function satisfying
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  $h(.,u)$ is increasing for all $u \in [0,1]$.
  is motivated by diffusion modelled by the p-Laplacian in the presence of advection. Consider the PDE

- $$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + g(u), \tag{2}$$

  where $k > 0$.

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  where $k > 0$.
- Looking for travelling waves in this case leads to

  $$(|u'|^{p-2}u')' + (c - ku)u' + f(u) = 0 \tag{3}$$

  with boundary conditions $u(-\infty) = 0, \ \ u(+\infty) = 1$, which was considered in the classical case $p = 2$ in Murray, Malaguti and Marcelli... Reduction to the first order gives (1) with $h(c, u) = c - ku$, and boundary conditions $y(0) = 0 = y(1)$ where $y = u'^p$.

Proposition

Suppose that $\lambda = \lim_{u \to 0} \frac{f(u)}{u^{q-1}}$ exists. If (1) has a positive solution in some interval $0 \leq u \leq \eta$ with $y(0) = 0$, then (i)

$$h(c,0) \geq (\lambda q)^{\frac{1}{q}} p^{\frac{1}{p}},$$

(ii) $\lim_{u \to 0} \frac{y(u)}{u^q}$ exists and coincides with one of the roots $w_{\pm}(c)$ of

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- It may be shown that (1) has a range of admissible speeds $[c^*, +\infty[$ and that c^* is also characterized in terms of the behaviour of trajectories at the origin.

Proposition

Let f be a function of type A satisfying $M = \sup\limits_{0<u\leq 1} \dfrac{f(u)}{u^{q-1}} < +\infty$.
Then there exists a constant $c^\star$, (depending on f, h and p) such that the boundary value problem $y' = q(h(c,u)y^{\frac{1}{p}} - f(u))$, $y(0) = 0 = y(1)$ admits a positive solution if and only if $c \geq c^\star$.
Moreover, the following estimate holds

$$\min_{u\in[0,1]} h(c^*,u) \leq M^{\frac{1}{q}} p^{\frac{1}{p}} q^{\frac{1}{q}} \tag{4}$$

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- Using lower solutions arguments, it is easy to see that c^* is a monotone function of h and f in the sense that

$$c^*(h_1,f_1) \le c^*(h_2,f_2) \quad \text{if} \quad h_1 \ge h_2,\ \ f_1 \le f_2.$$

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Proposition (Asymptotics)

Let c be an admissible speed of (1) and $y(u)$ be the unique solution of $y' = q(h(c,u)y^{\frac{1}{p}} - f(u)), \quad 0 \leq u \leq 1,$ $y(0) = 0 = y(1)$. Then
(i) if $c = c^*$ then $\lim_{u\to 0} \frac{y(u)}{u^q} = w_+(c)$
(ii) if $c > c^*$ then $\lim_{u\to 0} \frac{y(u)}{u^q} = w_-(c)$

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- **Example.** Take an example with advection: $h(c,u) = (c - ku)$ $(k > 0)$ and $f(u) = u^{q-1}(1-u)^{q-1}$ which is the analogue of the Fisher's reaction ($p = 2$).

- First note that the corresponding equation has a solution of the form $\alpha u^q(1-u)^q$ with

$$\alpha = \left(\frac{k}{2}\right)^q \quad \text{and} \quad c = \frac{k}{2} + \left(\frac{2}{k}\right)^{q-1}.$$

  Since the real function $x + \frac{1}{x^{q-1}}$ attains its minimum at $x = (q-1)^{1/q}$, we compute the corresponding values of k and c for which c is minimum:

$$k_0 = 2(q-1)^{1/q} \quad \text{and} \quad c_0 = q^{1/q}p^{1/p}.$$

  Claim: if $k \geq k_0$ then the critical speed is $c^* = \frac{k}{2} + \left(\frac{2}{k}\right)^{q-1}$.
  The analysis of $y(u)$ near $u = 0$ gives

$$\frac{y(u)}{u^q} = \left(\frac{k}{2}\right)^q (1-u)^q \to \left(\frac{k}{2}\right)^q \quad \text{as} \quad u \to 0$$

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  Since $\lambda = \lim_{u\to 0} \frac{f(u)}{u^{q-1}} = 1$, it follows that $w_+(\lambda) = \left(\frac{k}{2}\right)^q$.$\square$
- On the other hand, if $k = 0$ it is known that $c^* = q^{1/q}p^{1/p}$.
  By monotonicity of c^* with respect to h:
  $c^* = q^{1/q}p^{1/p} \quad \text{for} \quad 0 \leq k \leq k_0$.

- **Example.** The analogue of the Zeldovich equation in this context corresponds to $h(c,u) \equiv c$ $(k=0)$ and $f(u) = u^q(1-u)^{q-1}$. We find a solution of the form
$$y = \alpha u^q (1-u)^q$$
with $\alpha = \frac{1}{2}$ and $c = 2^{-1/q}$. Since $\lim_{u\to 0} \frac{f(u)}{u^{q-1}} = 0$ and $\lim_{u\to 0} \frac{y(u)}{u^q} = \frac{1}{2}$ we conclude from proposition 3 that in fact $c^* = 2^{-1/q}$.

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- For the analogue of the Zeldovich equation with the advection term $h(c,u) = c - ku$, where $k > 0$: there are solutions of the form $y(u) = \alpha u^q(1-u)^q$. with $\alpha = \alpha(k) > \left(\frac{k}{2}\right)^q$ being the only solution of the equation $kx^{1/p} - 2x = -1$ and $c = c(k) = \alpha^{1/q}$.
Notice that c depends continuously on k, is increasing and that $c \to 2^{-1/q}$ as $k \to 0$ and $c \to +\infty$ as $k \to +\infty$.
Finally, since $\lim_{u\to 0} \frac{f(u)}{u^{q-1}} = 0$ and $\lim_{u\to 0} \frac{y(u)}{u^q} = \alpha = w_+(0)$, proposition [Asymptotics] implies that $c^* = \alpha(k)^{-1/q}$ for all $k \geq 0$.

“RELATIVISTIC ACCELERATION” OPERATOR

- Now consider properties of the monotone solutions of the boundary value problem

$$(P(u'))' - cu' + f(u) = 0 \tag{5}$$

$$u(-\infty) = 0, u(+\infty) = 1. \tag{6}$$

which is motivated by the model case where $P(x) = \frac{x}{\sqrt{1-x^2}}$. This in turn corresponds to the problem of finding travelling waves for

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\frac{\partial u}{\partial x}}{\sqrt{1-(\frac{\partial u}{\partial x})^2}}\right] + f(u), \tag{7}$$

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- Assumptions on P
  1. $P \in C^1([0,1[)$
  2. $P(0) = 0$, and $P'(0) = 1$
  3. P is strictly increasing, $P(1^-) = +\infty$ and $\int_0^1 P(x)\,dx < \infty$.

- Since for monotone solutions there exists an inverse function $t(u)$, we may define $\phi(u) := P(u'(t(u)))$. Therefore $u'(t(u)) = P^{-1}(\phi(u))$ and setting

  $$v(u) := P^{-1}(\phi(u))$$

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- and denoting by $Q(x)$ the primitive of $x \mapsto P'(x)x$ such that $Q(0) = 0$, we obtain

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$$y(u) = Q(v(u)), \quad R = Q^{-1}.$$

- Then $y(u)$ satisfies the first order equation

$$y' = cR(y) - f(u) \tag{8}$$

where differentiation is done with respect to u. The boundary conditions for $u(t)$ in the real line translate into

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Remark

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Proposition

There exists a $1-1$ correspondence between solutions $u(t)$ of (5)-(6) (up to translation) and solutions $y(u)$ of (8)-(9) in such a way that (10) holds.

Proof Given a solution $y(u)$ of (8)-(9), the solution of the Cauhy problem (10) is defined in $]t_-, t_+[$, where

$$t_- = -\int_{1/2}^{1} \frac{du}{R(y(u))}, \quad t_+ = \int_{0}^{1/2} \frac{du}{R(y(u))}.$$

From the remark above we conclude that $t_- = -\infty$ and $t_+ = +\infty$.

Existence of solutions

We introduce the assumptions

(Hf) $f'(0)$ exists

(HP) Setting $E(y) := \int_0^y \frac{1}{R(x)}\, dx$ we have

$$\lim_{y \to 0} \frac{E(y)}{R(y)} = 1.$$

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Remark

$P(x) = \frac{x}{\sqrt{1-x^2}}$ satisfies the above assumption. In this case $E(y) = \sqrt{y(y+2)}$.

Proposition

If $y(u)$ is a solution of (8) such that $y(0) = 0$ and for some η $y(u) > 0$ for $0 < u < \eta$, then (i) $E(y)'(0) = \frac{d}{du}E(y(u))|_{u=0}$ exists and is a root x of $x^2 - cx + f'(0) = 0$. (ii) $c^2 \geq 4f'(0)$.

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Proposition

(i) The set of admissible speeds of (8)-(9) is an interval $[c^*, +\infty)$ where $c^* > 0$. (ii) Let $P(x) = \frac{x}{\sqrt{1-x^2}}$. If f satisfies for some $M > 0$ the estimate

$$f(u) \leq \frac{Mu}{\sqrt{1 + Mu^2}} \quad \forall u \in [0, 1] \qquad (*)$$

then $c^* \leq 2\sqrt{M}$.

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then $c^* \leq 2\sqrt{M}$.

- **Example** If $f(u) = u(1-u)/\sqrt{1+u^2}$, we have $c^* = 2$.

Asymptotics at $u = 0$

We set a new assumption, also satisfied by the model problem (R_1) $|R(t) - R(s)| \leq \frac{1}{\sqrt{2t}}|t - s|$ $\forall\, s \geq t \geq 0$.

Lemma

Assume $(R1)$. Consider the initial value problem

$$y'(u) = cR(y) - f(u), \quad y(0) = 0 \tag{11}$$

Let there $\eta > 0$, $0 < A < B$, $0 \leq a < b$, $0 < c_1 < c_2 < 2\sqrt{2}A$ be constants such that

$$a \leq \frac{f(u)}{u} \leq b, \quad \text{if} \quad 0 < u \leq \eta \tag{12}$$

$$2A^2 - c\sqrt{2}A + b < 0 < 2B^2 - c\sqrt{2}B + a \quad \forall c \in [c_1, c_2]. \tag{13}$$

Then, decreasing η if necessary: for $c \in [c_1, c_2]$ problem (11) has a unique solution y such that $A^2u^2 \leq y(u) \leq B^2u^2$ for $0 \leq u \leq \eta$. This solution depends continuously on c.

Proof is based on the Banach fixed point argument...

On the basis of this lemma it is easy to obtain the following proposition. We denote by $\lambda_-(c) \leq \lambda_+(c)$ the roots of the quadratic equation $x^2 - cx + f'(0) = 0$.

Proposition (ASYMPTOTICS)

Let c be an admissible speed of (8)-(9) and y be the corresponding solution of (8)-(9).

On the basis of this lemma it is easy to obtain the following proposition. We denote by $\lambda_{-}(c) \leq \lambda_{+}(c)$ the roots of the quadratic equation $x^2 - cx + f'(0) = 0$.

Proposition (ASYMPTOTICS)

Let c be an admissible speed of (8)-(9) and y be the corresponding solution of (8)-(9).

1. If $c = c^*$,

$$(E(y))'(0) = \lambda^{+}(c).$$

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Proposition (ASYMPTOTICS)

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2. If $c > c^*$,

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- Example: consider the analogue of Zeldovich equation, where

$$f(u) = \frac{u^2(1-u)}{\sqrt{1+\beta(u-u^2)^2}}.$$

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- Here the ansatz

$$y(y+2) = \beta(u-u^2)^2$$

or equivalently

$$y = -1 + \sqrt{1+\beta(u-u^2)^2}$$

yields a solution with $\beta = \frac{1}{2}$ and $c = \frac{1}{\sqrt{2}}$. Since $f'(0) = 0$ and $\lim_{u\to 0} \frac{y(u)}{u^2} = \frac{1}{4}$, we conclude from proposition [ASYMTOTICS] that this is the critical speed.

THE “CURVATURE” OPERATOR

- $$u_t = \left(\frac{u_x}{\sqrt{1+u_x^2}} \right)_x + f(u).$$

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1+u'^2}} \right)' - cu' + f(u) = 0 \\ \\ u(-\infty) = 0,\ u(+\infty) = 1; \end{cases} \tag{14}$$

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- where
$$P(v) = \frac{v}{\sqrt{1+v^2}}.$$

- $\phi(u) = P(u'(t(u)))$, $v(u) = P^{-1}(\phi(u))$, leads to $\frac{d}{du}Q(v(u)) - cv(u) + f(u) = 0$, where $Q(v)$ is a primitive of $vP'(v)$. Explicitly, $Q(v) = \int \frac{v}{\sqrt{1+v^2}^3}\,dv$, so that we can choose

$$Q(v) = 1 - \frac{1}{\sqrt{1+v^2}}, \quad (Q(0) = 0). \tag{16}$$

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- Setting $R = Q^{-1}$, $\quad R(y) = \frac{\sqrt{y(2-y)}}{1-y}, \quad 0 \leq y < 1$, this gives

$$y' = cR(y) - f(u),$$

Taking into account the boundary conditions, we thus want to study

$$\begin{cases} y' = c\frac{\sqrt{y(2-y)}}{1-y} - f(u) \\ y(0) = 0 = y(1), \end{cases} \tag{17}$$

Proposition (A range of admissible speeds)

Let f be of class A and assume that there exists $M > 0$ such that the following estimate holds:

$$f(u) \leq \frac{Mu}{\sqrt{1 - \min\{M, 1\}u^2}},$$

for every $u \in [0, 1]$. Then, for every

$$c \in [2\sqrt{M}, +\infty[\,,$$

problem (17) has a solution.

Reaction of “type C”

Another important form of the reaction term is the so called “type C”. Explicitly, we define

$$\mathcal{C} = \left\{ f \in C([0,1]) \;\middle|\; \begin{array}{l} f(0) = f(1) = 0 \text{ and there exists } \theta \in]0,1[\text{ s.t.} \\ f(u) < 0 \text{ for } u \in]0,\theta[\,, f(u) > 0 \text{ for } u \in]\theta, 1[\,. \end{array} \right\}$$

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Proposition

Let $f \in \mathcal{C}$. Then, there exists a positive admissible speed for f if and only if
the two following conditions simultaneously hold:

$$\int_0^1 f(u)\,du > 0, \quad \int_0^1 f^-(u)\,du < 1, \tag{18}$$

where $f^-(t) = \max\{-f(t), 0\}$. If this is true, the admissible speed is unique.

A VARIATIONAL PROPERTY OF THE CRITICAL SPEED (p-LAPLACian CASE)

- Back to the p-Laplacian case (without advection), consider travelling waves to reaction-diffusion equations driven by the one-dimensional p-Laplacian operator, namely

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u), \tag{19}$$

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- The relevant front wave profiles $u(x+ct)$ with speed c are given by the (monotone) solutions of the second order problem

$$(|u'|^{p-2}u')' - cu' + f(u) = 0 \tag{20}$$

satisfying the limit conditions

$$u(-\infty) = 0, \; u(+\infty) = 1 \tag{21}$$

- With q be the conjugate of p, that is $\frac{1}{p}+\frac{1}{q}=1$, the solutions of the parametric first order boundary value problem

$$y' = q(c\,{y_+}^{\frac{1}{p}} - f(u)), \quad 0 \le u \le 1, \quad y(0) = 0 = y(1), \ \ y > 0 \ \text{ in }] \tag{22}$$

yield the trajectories of solutions of (20)-(21) via the relationship

$$u' = y(u(t))^{1/p}.$$

This problem is a particular case of the one considered in the first part ($h(c,u) \equiv c$).

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  yield the trajectories of solutions of (20)-(21) via the relationship

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  This problem is a particular case of the one considered in the first part ($h(c,u) \equiv c$).
- We recall the natural assumptions for this problem.

(H_p) $\quad M_p := \sup_{0<u<1} \frac{f(u)}{u^{q-1}} < +\infty.$

(H_p') $\quad \mu := \lim_{u\to 0^+} \frac{f(u)}{u^{q-1}}$ exists, $0 \le \mu < +\infty.$

- There is a 1-1 correspondence between solutions of (20)-(21) (up to translation) taking values in $]0,1]$ and solutions of (22) that are strictly positive in $]0,1[$. These sets of solutions are nonempty provided (H_p) holds. Also, basic properties of the profiles and their speeds, now classical in the FKPP theory ($p = 2$), were extended to the p-Laplacian model (Enguiça, Gavioli, S.). In particular, if (H_p) holds, the set of admissible speeds – that is, values of the parameter c such that (22) has a solution – is an interval $[c^*, +\infty[$ where

$$\mu^{\frac{1}{q}} p^{\frac{1}{p}} q^{\frac{1}{q}} \leq c^* \leq M^{\frac{1}{q}} p^{\frac{1}{p}} q^{\frac{1}{q}} \tag{23}$$

(the first inequality being valid if the stronger $(H'p)$ holds). The minimum admissible value c^* of the parameter c is called *critical speed*.

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(the first inequality being valid if the stronger $(H'p)$ holds). The minimum admissible value c^* of the parameter c is called *critical speed*.
- For the case of linear diffusion ($p = 2$), variational caracterizations of the critical speed c^* are known (Arias, Campos, Pérez, S. (2004), and Benguria, Dépassier, Méndez (2004)). Let us exhibit a variational property of c^* in the framework of the p-Laplacian.

Remark

We recall the role played by functions of *type B*. A function $f : [0,1] \to \mathbb{R}$ is said to be of type B if it is continuous and there exists $\delta \in]0,1[$ such that $f(s) = 0$ if $0 \leq s \leq \delta$ or $s = 1$, and $f(s) > 0$ if $\delta < s < 1$.

It is known that if f is of type B there exists exactly one admissible speed c^* of (20)-(21), that is, (22) has a positive solution for exactly this value of the parameter c. Moreover, if f_n is a nondecreasing sequence of functions of type B and $\lim_{n\to\infty} f_n(x) = f(x)$, then with obvious notation $\lim_{n\to\infty} c^*(f_n) = c^*(f)$. This is used in the proof of the main result.

Some equivalent boundary value problems

For convenience, we start by considering a different model, with homogeneity of degree $p-1$ in the derivatives. Consider the problem

$$(u'^{p-1})' - c^{p-1}u'^{p-1} + f(u) = 0. \tag{24}$$

$$u(-\infty) = 0,\ u(+\infty) = 1 \tag{25}$$

which, by the way, may be seen as the search for travelling waves of the form $u(x+ct)$ for the quasilinear parabolic equation in one spacial dimension.

$$\frac{\partial(u^{p-1})}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u), \tag{26}$$

The homogeneity appearing in the quasilinear term of (24) is used in the following way. If we perform the change of variable $s = e^{kt}$ with $k > 0$, and define $v(s) = u(t)$, this problem is seen to be equivalent to the following boundary value problem in $[0, +\infty[$

$$(v'^{p-1})' + \frac{1}{k^p}\frac{f(v(s))}{s^p} = 0 \tag{27}$$

$$v(0) = 0,\ v(+\infty) = 1,\ \ v' > 0 \tag{28}$$

provided

$$c^{p-1} = k(p-1).$$

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- so that $\psi = \varphi^{p}$ solves

$$\psi' = q(c^{p-1}\psi^{\frac{1}{q}} - f(u)) \tag{29}$$

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- Acording to what has been recalled in the Introduction, (29) -(30) has solutions provided that

$$(H_q) \qquad M_q := \sup_{0<u<1} \frac{f(u)}{u^{p-1}} < +\infty.$$

Rewriting this as $\psi' = p(c^{p-1}\frac{q}{p}\psi^{\frac{1}{q}} - \frac{q}{p}f(u))$ we assert that the set of admissible speeds c is an interval $[c^*, +\infty[$ where ${c^*}^{p-1} \leq M^{\frac{1}{p}} q$. If, in addition, we assume the stronger assumption

(H_q') $\quad \nu := \lim_{u \to 0^+} \frac{f(u)}{u^{p-1}}$ exists, $0 \leq \nu < +\infty$

then we also have the lower estimate

$$ {c^*}^{p-1} \geq \nu^{\frac{1}{p}} q. \tag{31} $$

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Let f be of type A and (H_q) hold, or let f be of type B. Then the following are equivalent

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- - (27) -(28)**[2nd order, half-line]** with $k = \frac{c^{p-1}}{p-1}$ has a (concave) solution with $v' > 0$ in some interval $]0, \beta[$, and $u(\beta^-) = 1$.

Remark

If f is of type B, **[2nd order, half-line]** is solvable only for $k = k^* := \frac{(c^*)^{p-1}}{p-1}$.

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Proposition

Suppose that ψ solves **[1st order]** with $c > c^*$. Then

$$\lim_{u \to 0} \frac{\psi(u)}{u^p} < \left(\frac{c^{p-1}}{p} \right)^p .$$

A constrained minimum problem

- We relate **[2nd order, real line]** with the nonlinear singular boundary value problem

$$(v'^{p-1})' + \lambda \frac{f(v(s))}{s^p} = 0, \quad v(0) = 0,\ v(+\infty) = 1,\ \ v' > 0 \tag{32}$$

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- Extend f with zero value outside $[0,1]$ and set

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- In addition we consider the space of functions

$$E = \{v \in AC([0,+\infty[,\mathbb{R}) \,|\, v' \in L^p(0,+\infty)\ \ ,\ \ v(0) = 0.\}$$

and the following real functionals on E

$$J(v) = \frac{1}{p}\int_0^{+\infty} |v'(s)|^p\,ds, \quad \Gamma(v) = \int_0^{+\infty} \frac{F(v(s))}{s^p}\,ds.$$

We remark that (H_q) is sufficient for Γ to be well defined and C^1 in E, by Hardy's inequality.

Set

$$\theta = \inf_{v \in E \setminus 0} \frac{J(v)}{\Gamma(v)}. \tag{33}$$

Theorem

Let f be of type B, or of type A and (H_q') holds. We have $\nu q^p \theta \leq 1$. If $\nu q^p \theta < 1$ then the inf in (33) is attained. In any case $\theta^{1/p} = \frac{p-1}{c^{*p-1}}$ where c^* is the least admissible value of c so that (29)-(30)has solutions.

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- In the previous section we have given a variational characterization of the least value c such that a certain parametric first order problem is solvable. By interchanging p and q, reading carefully the first order equation that corresponds to our original problem, we obtain:

Theorem

Let f be a function of type A and assume (H'_p). Define

$$\gamma = \inf_{v \in F \setminus 0} \frac{\frac{1}{q} \int_0^{+\infty} |v'(s)|^q \, ds}{\int_0^{+\infty} \frac{F(v(s))}{s^q} \, ds}.$$

Then the critical speed for (34) is the number c^* given by

$$\gamma = \frac{q}{p {c^*}^q}.$$

Moreover γ is attained if $\mu p^q \gamma < 1$.