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When a thin periodic layer meets corners : asymptotic analysis of a singular Helmholtz problem.

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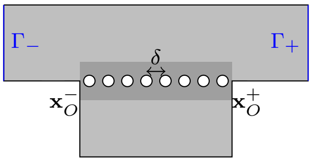
Model problem

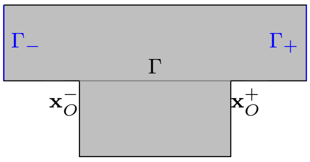
We consider a domain Ω^δ containing a finite periodic layer of size δ, and we are interested in the solution of the Helmholtz problem with a source term $g_\pm$ corresponding to an incident field coming from the left:

$$\left\{\begin{array}{rcll} -\Delta u^\delta - (k^\delta)^2 u^\delta &=& 0, & \text{in } \Omega^\delta, \\ -\nabla u^\delta \cdot \mathbf{n} - \imath k_0 u^\delta &=& g_\pm, & \text{on } \Gamma_\pm, \\ -\nabla u^\delta \cdot \mathbf{n} &=& 0, & \text{on } \partial\Omega^\delta \setminus \Gamma_\pm, \end{array}\right.$$

where the wavenumber $k^\delta(\mathbf{x}) = \widehat{k}(\mathbf{x}/\delta)$ differs from a constant value k_0 by a δ-periodic function w.r.t. x_1 in $(-L,L)\times(-\delta,\delta)$.



Domain of interest Ω^δ.



Limit domain Ω^0 of Ω^δ as $\delta \to 0$.

Asymptotic expansion

Based on the method of matched asymptotic expansions, we decompose the domain Ω^δ into three regions.

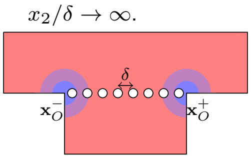
- For the two near-field regions, we seek for u^δ under the form

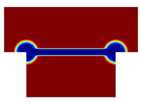
$$u^\delta(\mathbf{x}) = \sum_{(n,q)\in\mathbb{N}^2} \delta^{2n/3+q} U^\delta_{n,q,\pm}\left(\frac{\mathbf{x}-\mathbf{x}_O^\pm}{\delta}\right).$$

- For the far-field region, we search for u^δ under the form

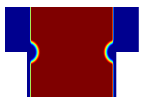
$$u^\delta(\mathbf{x}) = \sum_{(n,q)\in\mathbb{N}^2} \delta^{2n/3+q}\left(\chi^\delta_{\text{macro}} u^\delta_{n,q} + \chi^\delta_{\text{layer}} \Pi^\delta_{n,q}\left(x_1, \frac{\mathbf{x}}{\delta}\right)\right)$$

where the periodic layer correctors $\Pi^\delta_{n,q}$ decay exponentially towards $x_2/\delta \to \infty$.





$\chi^\delta_{\text{macro}}$

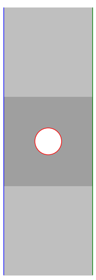


$\chi^\delta_{\text{layer}}$

Periodic layer problems

First we insert the far-field ansatz in the Helmholtz equation. Separating the behavior for the macroscopic scale $\mathbf{x}$ and the microscopic scale $\mathbf{X} = \mathbf{x}/\delta$ gives for $u_{n,q}$ the homogeneous Helmholtz equation $-\Delta u_{n,q} - k_0^2 u_{n,q} = 0$ for $x_2 \neq 0$ with appropriate boundary conditions. The periodic boundary layer corrector $\Pi_{n,q}$ satisfies the following problem

$$\begin{array}{rcll} -\Delta_{\mathbf{X}} \Pi_{n,q} &=& G_{n,q}, & \text{in } (-L,L)\times\mathcal{B} \\ \nabla\Pi_{n,q}\cdot\mathbf{n} &=& 0, & \text{on } (-L,L)\times\partial\mathcal{B}_{\text{int}} \\ \Pi_{n,q}(x_1, 0, X_2) &=& \Pi_{n,q}(x_1, 1, X_2) & \\ \partial_{X_1}\Pi_{n,q}(x_1, 0, X_2) &=& \partial_{X_1}\Pi_{n,q}(x_1, 1, X_2) & \\ X_2^\alpha \Pi_{n,q}(x_1, X_1, X_2) &\to& 0, & \alpha\in\mathbb{N}, \pm X_2 \to \infty \end{array}$$



Studying this family of problem gives the jump conditions

$$[u_{n,0}]_\Gamma(x_1) = 0, \quad [\partial_{x_2} u_{n,0}]_\Gamma(x_1) = 0$$

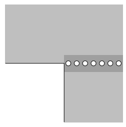
for any $n\in\mathbb{N}$, and (for symmetric $\mathcal{B}$)

$$[u_{n,1}]_\Gamma(x_1) = C_1 \langle \partial_{x_2} u_{n,0}\rangle_\Gamma(x_1),$$
$$[\partial_{x_2} u_{n,1}]_\Gamma(x_1) = \left(C_2 + C_3\partial^2_{x_1}\right)\langle u_{n,0}\rangle_\Gamma(x_1).$$

Study of the near-field problems

Writing the matching conditions between the far-field ansatz and the near-field ansatz exhibits this family of singular problems for $n\in\mathbb{N}$:

$$\begin{array}{l} -\Delta_{\mathbf{X}} S_n = 0, \quad \text{in } \widehat{\Omega}^- \\ -\nabla S_n \cdot \mathbf{n} = 0, \quad \text{on } \partial\widehat{\Omega}^- \\ S_n(R,\theta) \sim R^{\frac{2n}{3}} w_{n,0}(\theta), \quad R\to\infty \end{array}$$



with

$$w_{n,0}(\theta) = \cos\frac{2n}{3}(\theta-\pi)$$

This problem cannot be solved in the classical Beppo-Levi space since the desired behavior towards infinity is an increasing behavior. We need then to introduce weighted Sobolev spaces $\mathfrak{V}^\ell_{\beta,\gamma}(\widehat{\Omega}^-)$ embedded with the norm

$$\|v\|_{\mathfrak{V}^\ell_{\beta,\gamma}(\widehat{\Omega}^-)} = \sum_{p=0}^{\ell}\left\|(1+R)^{\beta-\gamma-\delta_{p,0}}\rho^{\gamma-\ell+p+\delta_{p,0}}\nabla^p v\right\|_{L^2(\widehat{\Omega}^-)}.$$

For the classical Kondratiev theory [1], the weight ρ is $\rho = 1+R$. Here, in the spirit of the works of Nazarov [2], we choose

$$\rho = 1 + R\,|\theta|$$

that allows different weight behaviors close to the infinite periodic layer, and away from it. Studying this problem allows us then to give an expansion of S_n towards $R\to\infty$ and for $\theta\neq 0$

$$S_n = \sum_{p=0}^{\infty} R^{\frac{2n}{3}-p} w_{n,p}(\ln R, \theta) + \sum_{m=1}^{\infty}\mathscr{L}_{-m}(S_n)\sum_{p=0}^{\infty} R^{-\frac{2m}{3}-p} w_{-m,p}(\ln R,\theta)$$

Singular behavior of macroscopic terms

Using the information of the periodic layer problems and the matching conditions, we can deduce iteratively the behavior of the near-field terms $u_{n,q}$. For the three first non-trivial terms, we obtain

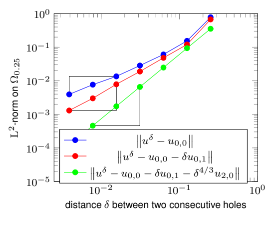
$$\begin{array}{ll} u_{0,0}(\mathbf{x}) \to c_{0,0}, & \mathbf{x}\to\mathbf{x}_O^-, \\ u_{0,1}(\mathbf{x}) \sim |\mathbf{x}-\mathbf{x}_O^-|^{\frac{2}{3}-1}, & \mathbf{x}\to\mathbf{x}_O^-, \\ u_{2,0}(\mathbf{x}) \sim |\mathbf{x}-\mathbf{x}_O^-|^{-\frac{2}{3}}, & \mathbf{x}\to\mathbf{x}_O^-. \end{array}$$

Error estimates

Theorem. Let $a < L$, and let Ω_a be the domain in which we cut a rectangle centered at $(0,0)$ of size $(2L+2a)\times 2a$. Then, we have for δ small enough

$$\Big\|u^\delta - \sum_{(n,q)\in I}\delta^{\frac{2n}{3}+q}u_{n,q}\Big\|_{L^2(\Omega_a)} \leqslant C\delta^2\ln\delta$$

where $I = \{(0,0),(0,1),(2,0)\}$.



References

[1] V. A. Kozlov, V. G. Mazya and J. Rossmann, Elliptic boundary value problems in domains with point singularities. *AMS*, Mathematical Surveys and Monographs (52), 1997, pp:x+414.

[2] S. A. Nazarov., The Neumann problem in angular domains with periodic and parabolic perturbations of the boundary. *Tr. Mosk. Mat. Obs.*, 69:182–241, 2008.

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