EXISTENCE AND REGULARITY OF SOLUTIONS TO OPTIMAL PARTITION PROBLEMS INVOLVING LAPLACIAN EIGENVALUES

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Calculus of Variations and Applications
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Joint work with Miguel Ramos and Susanna Terracini
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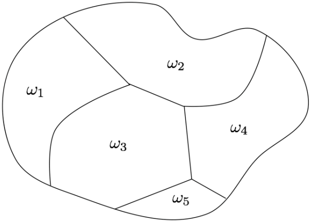
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Take $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}$.

Denote $\lambda_{k_i}(\omega)$ as being the k_i–th eigenvalue of $(-\Delta, H_0^1(\omega))$.

$$\inf\left\{\sum_{i=1}^{m} \lambda_{k_i}(\omega_i) : \ \omega_1, \ldots, \omega_m \subseteq \Omega \text{ open sets}, \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\right\}$$



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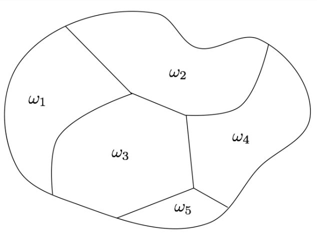

Main goals:

- Existence of an optimal partition;
- Study of the regularity of the optimal partition and associated eigenfunctions.

- Class of admissible sets: $\mathcal{A}(\Omega)$
- Cost Functional: $\Phi : \mathcal{A}(\Omega)^m \to \mathbb{R}$

Minimization problem:

$$\inf \{\Phi(\omega_1, \ldots, \omega_m) : \ \omega_i \in \mathcal{A}(\Omega), \ \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\}$$

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Cases in which we have classical solutions:

- geometric constraints on the admissible domains (convexity, number of connected components, ...)
- Φ has good monotonicity properties.

Reference: the book [Bucur and Buttazzo (2005)]

Theorem (Ramos, T., Terracini)

The optimal partition problem in consideration admits an open regular solution $(\omega_1, \ldots, \omega_m)$.

$$\inf \left\{ \sum_{i=1}^{m} \lambda_{k_i}(\omega_i) : \ \omega_1, \ldots, \omega_m \subseteq \Omega \text{ open sets}, \omega_i \cap \omega_j = \emptyset \ \forall i \neq j \right\}$$

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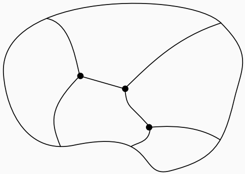

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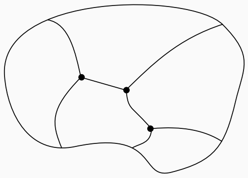

1. denoting $\Gamma = \Omega \setminus \bigcup_{i=1}^{m} \omega_i$, it holds $\mathscr{H}_{\dim}(\Gamma) \leq N - 1$;
2. we can write $\Gamma = \mathcal{R} \cup \mathcal{S}$, with:
   - $\mathscr{H}_{\dim}(\mathcal{S}) \leq N - 2$.
   - $\mathcal{R}$ is a collection of hypersurfaces of class $C^{1,\alpha}$, each one separating two different elements of the partition;

Theorem (cont.)

For each $i = 1, \ldots, m$ *there exists* $1 \leq l_i \leq k_i$ *and*

- $u_1^i, \ldots, u_{l_i}^i$ *eigenfunctions associated to the eigenvalue* $\lambda_{k_i}(\omega_i)$*;*
- *coefficients* $a_1^i, \ldots, a_{l_i}^i > 0$

such that

Theorem (cont.)

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such that

- $u_1^i, \dots, u_{l_i}^i$ *are* ***Lipschitz continuous****;*
- ***Extremality condition on the regular part of the boundary****:* *given* $x_0 \in \mathcal{R}$*, denoting by* ω_i *and* ω_j *the two adjacent sets of the partition at* x_0*,*

$$\lim_{\substack{x \to x_0 \\ x \in \omega_i}} \sum_{n=1}^{l_i} a_n^i |\nabla u_n^i(x)|^2 = \lim_{\substack{x \to x_0 \\ x \in \omega_j}} \sum_{n=1}^{l_j} a_n^j |\nabla u_n^j(x)|^2 \neq 0.$$

Theorem (cont.)

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The question of whose eigenfunctions and coefficients interfere is not obvious at all, and it is not given initially by the problem: it comes out from our proof.

Plan for the rest of the talk:

1. References
2. The first step of our strategy: a double approximation result

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First simplification from now on: $k_i \equiv k$

$$\Phi(\omega_1, \ldots, \omega_m) = \sum_{i=1}^{m} \lambda_k(\omega_i)$$

Second simplification from now on: partitions with two sets

$$\Phi(\omega_1, \omega_2) = \lambda_k(\omega_1) + \lambda_k(\omega_2)$$

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Fix $k \in \mathbb{N}$.

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left(\lambda_k(\omega_1) + \lambda_k(\omega_2)\right)$$

with

$$\mathcal{P}_2(\Omega) = \left\{(\omega_1,\omega_2) : \; \omega_1,\omega_2 \subset \Omega \text{ open}, \; \omega_1 \cap \omega_2 = \emptyset\right\}.$$

1. References

Existence results for any k:

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_k(\omega_1)+\lambda_k(\omega_2))$$

- [Bucur, Buttazzo, Henrot, Adv. Math. Sci. Appl. (1998)]
  - existence in the class of *quasi-open* sets
  - γ and weak γ–convergence, direct methods
- [Bourdin, Bucur, Oudet, SIAM J. Sci. Comp. (2009)]
  - existence in the class of *open* sets for $N = 2$
  - penalization with partition of the unity functions

Existence and Regularity of Open Partitions
Sum of first eigenvalues: $k = 1$

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left(\lambda_1(\omega_1) + \lambda_1(\omega_2)\right)$$

1st approach

- [Conti, Terracini, Verzini, CVPDE (2005)]
- [Caffarelli, F.H. Lin, J. Sci. Comp. (2007)]

$$\inf\left\{\int_\Omega (|\nabla u|^2 + |\nabla v|^2) : \ u, v \in H_0^1(\Omega), \ \int_\Omega u^2 = \int_\Omega v^2 = 1, \ u\cdot v \equiv 0\right\}$$

2nd approach. Eigenfunctions as limiting profiles of solutions to singularly perturbed systems with competitive interactioin

- [Chang, Lin, Lin Lin, Phys. D (2004)]
- [Conti, Terracini, Verzini, CVPDE (2005)]

$$\begin{cases} -\Delta u = \lambda_\beta u - \beta u v^2 \\ -\Delta v = \mu_\beta v - \beta u^2 v \\ u, v \in H_0^1(\Omega), \quad \int_\Omega u^2 = \int_\Omega v^2 = 1 \end{cases}$$

$$\Big\downarrow \ \beta \to +\infty$$

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$$E(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) + \frac{\beta}{2}\int_\Omega u^2 v^2 \quad \text{with} \quad \int u^2 = \int v^2 = 1$$

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$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_1(\omega_1) + \lambda_1(\omega_2))$$

- [T., Terracini AIHP (2012)]. Sum of second eigenvalues: $k = 2$.

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_2(\omega_1) + \lambda_2(\omega_2))$$

2. The first step of our strategy

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Initial Motivation: to find a strategy not relying on the multiplicity of the optimal sets

- double approximation problem
- obtain approximating solutions through a minimization approach

Basic facts:

- given $a_1, \ldots, a_k \geq 0$,

$$(a_1^p + \ldots + a_k^p)^{1/p} \to \max\{a_1, \ldots, a_k\} \quad \text{as } p \to +\infty$$

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$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_k(\omega_1) + \lambda_k(\omega_2))$$

$$\Big\uparrow \; p \to +\infty$$

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left\{ \left(\sum_{j=1}^{k} \lambda_j(\omega_1)^p \right)^{1/p} + \left(\sum_{j=1}^{k} \lambda_j(\omega_2)^p \right)^{1/p} \right\}$$

Approach this last problem, for each $p \in \mathbb{N}$, with a singularly perturbed system

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$$\Big\uparrow \; \beta \to \infty$$

Minimizer of a singularly perturbed system where competition occurs between groups of components $(u_1, \ldots, u_k)$ vs $(v_1, \ldots, v_k)$.

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$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left\{ \left(\sum_{j=1}^{k} \lambda_j(\omega_1)^p \right)^{1/p} + \left(\sum_{j=1}^{k} \lambda_j(\omega_2)^p \right)^{1/p} \right\}$$

$$\Big\uparrow \; \beta \to +\infty$$

$$\begin{cases} -a_{i,\beta}\Delta u_i = \sum_{j=1}^{k} \lambda_{ij,\beta} u_j - \beta u_i (v_1^2 + \ldots + v_k^2) \\ -b_{i,\beta}\Delta v_i = \sum_{j=1}^{k} \mu_{ij,\beta} v_j - \beta v_i (u_1^2 + \ldots + u_k^2) \end{cases}$$

with

$$\int_\Omega u_i u_j = \int_\Omega v_i v_j = \delta_{ij} \quad \forall i,j, \qquad \int_\Omega \nabla u_i \cdot \nabla u_j = \int_\Omega \nabla v_i \cdot \nabla v_j = 0 \quad \forall i \neq j$$


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