EXISTENCE AND REGULARITY OF SOLUTIONS TO OPTIMAL PARTITION PROBLEMS INVOLVING LAPLACIAN EIGENVALUES

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Calculus of Variations and Applications
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Joint work with Miguel Ramos and Susanna Terracini

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Take $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}$.

Denote $\lambda_{k_i}(\omega)$ as being the $k_i$–th eigenvalue of $(-\Delta, H^1_0(\omega))$.

$$\inf \left\{ \sum_{i=1}^{m} \lambda_{k_i}(\omega_i) : \omega_1, \ldots, \omega_m \subseteq \Omega \text{ open sets, } \omega_i \cap \omega_j = \emptyset \ \forall i \neq j \right\}$$

Main goals:

• Existence of an optimal partition;
• Study of the regularity of the optimal partition and associated eigenfunctions.
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• Class of admissible sets: \( A(\Omega) \)
• Cost Functional: \( \Phi : A(\Omega)^m \to \mathbb{R} \)

Minimization problem:

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Cases in which we have classical solutions:

• geometric constraints on the admissible domains (convexity, number of connected components, ...)
• $\Phi$ has good monotonicity properties.

Reference: the book [Bucur and Buttazzo (2005)]
Our main result

Theorem (Ramos, T., Terracini)

The optimal partition problem in consideration admits an open regular solution $(\omega_1, \ldots, \omega_m)$.

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Here, regularity means that:

1. denoting \(\Gamma = \Omega \setminus \bigcup_{i=1}^{m} \omega_i\), it holds \(\mathcal{H}_{\text{dim}}(\Gamma) \leq N - 1\);
2. we can write \(\Gamma = \mathcal{R} \cup \mathcal{S}\), with:
   - \(\mathcal{H}_{\text{dim}}(\mathcal{S}) \leq N - 2\).
   - \(\mathcal{R}\) is a collection of hypersurfaces of class \(C^{1,\alpha}\), each one separating two different elements of the partition;
Theorem (cont.)

For each $i = 1, \ldots, m$ there exists $1 \leq l_i \leq k_i$ and

- $u^i_1, \ldots, u^i_{l_i}$ eigenfunctions associated to the eigenvalue $\lambda_{k_i}(\omega_i)$;
- coefficients $a^i_1, \ldots, a^i_{l_i} > 0$

such that
Theorem (cont.)

For each \( i = 1, \ldots, m \) there exists \( 1 \leq l_i \leq k_i \) and

- \( u_1^i, \ldots, u_{l_i}^i \) eigenfunctions associated to the eigenvalue \( \lambda_{k_i}(\omega_i) \);
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such that

- \( u_1^i, \ldots, u_{l_i}^i \) are Lipschitz continuous;
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such that

- $u^i_1, \ldots, u^i_{l_i}$ are **Lipschitz continuous**;
- **Extremality condition on the regular part of the boundary:**
  
  given $x_0 \in \mathbb{R}$, denoting by $\omega_i$ and $\omega_j$ the two adjacent sets of the partition at $x_0$,
  
  $$\lim_{x \to x_0} \sum_{x \in \omega_i} a^i_n |\nabla u^i_n (x)|^2 = \lim_{x \to x_0} \sum_{x \in \omega_j} a^j_n |\nabla u^j_n (x)|^2 \neq 0.$$
Theorem (cont.)

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- \( u_1^i, \ldots, u_{l_i}^i \) are Lipschitz continuous;
- **Extremality condition on the regular part of the boundary:** given \( x_0 \in \mathbb{R} \), denoting by \( \omega_i \) and \( \omega_j \) the two adjacent sets of the partition at \( x_0 \),

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\lim_{x \to x_0} \sum_{n=1}^{l_i} a_n^i |\nabla u_n^i(x)|^2 = \lim_{x \to x_0} \sum_{n=1}^{l_j} a_n^j |\nabla u_n^j(x)|^2 \neq 0.
\]

The question of whose eigenfunctions and coefficients interfere is not obvious at all, and it is not given initially by the problem: it comes out from our proof.
Plan for the rest of the talk:

1. References
2. The first step of our strategy: a double approximation result
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2. The first step of our strategy: a double approximation result

First simplification from now on: $k_i \equiv k$

$$
\Phi(\omega_1, \ldots, \omega_m) = \sum_{i=1}^{m} \lambda_k(\omega_i)
$$

Second simplification from now on: partitions with two sets

$$
\Phi(\omega_1, \omega_2) = \lambda_k(\omega_1) + \lambda_k(\omega_2)
$$
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Fix $k \in \mathbb{N}$.

\[
\inf_{(\omega_1,\omega_2) \in \mathcal{P}_2(\Omega)} (\lambda_k(\omega_1) + \lambda_k(\omega_2))
\]

with

\[
\mathcal{P}_2(\Omega) = \{(\omega_1,\omega_2) : \omega_1,\omega_2 \subset \Omega \text{ open, } \omega_1 \cap \omega_2 = \emptyset\}.
\]
1. References
Existence results for any $k$:

$$\inf_{(\omega_1, \omega_2) \in \mathcal{P}_2(\Omega)} (\lambda_k(\omega_1) + \lambda_k(\omega_2))$$

  - existence in the class of quasi-open sets
  - $\gamma$ and weak $\gamma$–convergence, direct methods

- [Bourdin, Bucur, Oudet, SIAM J. Sci. Comp. (2009)]
  - existence in the class of open sets for $N = 2$
  - penalization with partition of the unity functions
Existence and Regularity of Open Partitions
Sum of first eigenvalues: $k = 1$

$$\inf_{(\omega_1, \omega_2) \in \mathcal{P}_2(\Omega)} (\lambda_1(\omega_1) + \lambda_1(\omega_2))$$

1st approach

- [Conti, Terracini, Verzini, CVPDE (2005)]

$$\inf \left\{ \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) : u, v \in H^1_0(\Omega), \int_{\Omega} u^2 = \int_{\Omega} v^2 = 1, u \cdot v \equiv 0 \right\}$$
2nd approach. Eigenfunctions as limiting profiles of solutions to singularly perturbed systems with competitive interaction

- [Chang, Lin, Lin Lin, Phys. D (2004)]
- [Conti, Terracini, Verzini, CVPDE (2005)]

\[
\begin{aligned}
-\Delta u &= \lambda_\beta u - \beta uv^2 \\
-\Delta v &= \mu_\beta v - \beta u^2v \\
\end{aligned}
\]

\[ u, v \in H^1_0(\Omega), \quad \int_\Omega u^2 = \int_\Omega v^2 = 1 \]

\[ \beta \to +\infty \]

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\begin{cases}
-\Delta u = \lambda \beta u - \beta uv^2 \\
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\]

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\[E(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2) + \frac{\beta}{2} \int_\Omega u^2 v^2 \quad \text{with} \quad \int u^2 = \int v^2 = 1\]
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- [T., Terracini AIHP (2012)]. Sum of second eigenvalues: \(k = 2\).
2. The first step of our strategy
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**Initial Motivation:** to find a strategy not relying on the multiplicity of the optimal sets

- double approximation problem
- obtain approximating solutions through a minimization approach
Basic facts:

- given $a_1, \ldots, a_k \geq 0$,

\[
(a_1^p + \ldots + a_k^p)^{1/p} \to \max \{a_1, \ldots, a_k\} \quad \text{as } p \to +\infty
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- $\lambda_k(\omega) = \max\{\lambda_1(\omega), \ldots, \lambda_k(\omega)\}$
Approach this last problem, for each $p \in \mathbb{N}$, with a singularly perturbed system

$$\inf_{(\omega_1, \omega_2) \in \mathcal{P}_2(\Omega)} \left\{ \left( \sum_{j=1}^{k} \lambda_j (\omega_1)^p \right)^{1/p} + \left( \sum_{j=1}^{k} \lambda_j (\omega_2)^p \right)^{1/p} \right\}$$

Minimizer of a singularly perturbed system where competition occurs between groups of components $(u_1, \ldots, u_k)$ vs $(v_1, \ldots, v_k)$. 

$\beta \to \infty$
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$$\uparrow \quad \beta \to +\infty$$

$$\begin{cases}
-a_{i,\beta} \Delta u_i = \sum_{j=1}^{k} \lambda_{ij,\beta} u_j - \beta u_i (v_1^2 + \ldots + v_k^2) \\
b_{i,\beta} \Delta v_i = \sum_{j=1}^{k} \mu_{ij,\beta} v_j - \beta v_i (u_1^2 + \ldots + u_k^2)
\end{cases}$$

with

$$\int_{\Omega} u_i u_j = \int_{\Omega} v_i v_j = \delta_{ij} \quad \forall i, j, \quad \int_{\Omega} \nabla u_i \cdot \nabla u_j = \int_{\Omega} \nabla v_i \cdot \nabla v_j = 0 \quad \forall i \neq j$$
TO LUÍSA MASCARENHAS