On some regularity results for elliptic boundary value problems.

H. Beirão da Veiga

Department of Mathematics
Pisa University

International Workshop on Calculus of Variations and its Applications
On the occasion of Luísa Mascarenhas birthday
Universidade Nova de Lisboa (Campus de Caparica), Portugal,
December, 17-19, 2015.

Introduction

Consider second order, uniformly elliptic operators

$$\boldsymbol{L} = \sum_{1}^{n} a_{i\,j}(x)\, \partial_i\, \partial_j\,. \tag{0.1}$$

Assume that one looks for *minimal assumptions* on f which guarantee that the solutions u to the problem

$$\begin{cases} \boldsymbol{L}\, u = f \quad \text{in} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \Gamma\,, \end{cases} \tag{0.2}$$

satisfy $\nabla^2\, u \in C(\overline{\Omega})$ (classical solutions). A Hölder continuity assumption on f is unnecessarily restrictive. On the other hand, continuity of f is not sufficient to guarantee continuity of $\nabla^2\, u$. This situation led to consider a Banach space $C_*(\overline{\Omega})$, $C^{0,\,\lambda}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset C(\overline{\Omega})$, for which

$$f \in C_*(\overline{\Omega}) \quad \Longrightarrow \quad u \in C^2(\overline{\Omega})\,.$$

We consider the (Dini's type) semi-norm

$$[f]_* = [f]_{*,R} \equiv \int_0^R \omega_f(r)\,\frac{dr}{r}\,, \tag{0.3}$$

and the functional space

$$C_*(\overline{\Omega}) \equiv \{\, f \in C(\overline{\Omega}) : [f]_* < \infty \,\} \tag{0.4}$$

normalized by $\| f \|_* = [f]_* + \| f \|$.
$C_*(\overline{\Omega})$ is a Banach space, the embedding $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact, and the set $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$. An "extension theorem" holds.

For data in $C_*(\overline{\Omega})$ no significant additional regularity, besides mere continuity, seems to be enjoyed by $\nabla^2 u$. On the contrary, in the case of Hölder continuity, full regularity is restored ($\nabla^2 u$ and f have the same regularity). So, we are in the presence of two totally opposite behaviors.

The above picture leads to consider general data spaces $D_{\overline{\omega}}(\overline{\Omega})$, characterized by a given *modulus of continuity* function $\overline{\omega}(r)$, contained between $Lip(\overline{\Omega})$ and $C_*(\overline{\Omega})$.

The spaces $D_\omega(\overline{\Omega})$.

We consider real, *continuous*, *non-decreasing* functions $\omega(r)$, defined for $0 \leq r < R$, for some $R > 0$. Furthermore, $\omega(0) = 0$, and $\omega(0) > 0$ for $r > 0$.
For $f \in C(\overline{\Omega})$ and $r > 0$, we define the modulus of continuity

$$\omega_f(r) \equiv \sup_{x,\, y \in \Omega\, ;\, 0 < |x - y| \leq r} |\, f(x) - f(y)\, | \,, \tag{0.5}$$

the semi-norm

$$[f]_\omega = \sup_{0 < r < R} \frac{\omega_f(r)}{\omega(r)} \,, \tag{0.6}$$

and the linear space

$$D_\omega(\overline{\Omega}) = \{\, f \in C(\overline{\Omega}) : [f]_\omega < \infty \,\} \,,$$

normalized by $\|f\|_\omega = [f]_\omega + \|f\|$.

Theorem

$D_\omega(\overline{\Omega})$ is a Banach space.

Theorem

The embedding $D_\omega(\overline{\Omega}) \subset D_{\omega_1}(\overline{\Omega})$ is compact if

$$\lim_{r\to 0} \frac{\omega(r)}{\omega_1(r)} = 0\,. \tag{0.7}$$

Theorem

If (0.7) holds then $D_{\omega_1}(\overline{\Omega})$ is not dense in $D_\omega(\overline{\Omega})$.

Theorem

Set $\Omega_\delta \equiv \{\, x : \, dist(x, \Omega\,) < \delta\,\}$. There is a $\delta > 0$ such that the following holds. There is a linear continuous extension map T from $C(\overline{\Omega})$ to $C(\overline{\Omega}_\delta)$, and from $D_\omega(\overline{\Omega})$ to $D_\omega(\overline{\Omega}_\delta)$, such that $T\,f(x) = f(x)$, for each $x \in \overline{\Omega}$.

Spaces $D_{\overline{\omega}}(\overline{\Omega})$ and $D_{\widehat{\omega}}(\overline{\Omega})$. The regularity theorems.

In dealing with $D_{\omega}(\overline{\Omega})$ in the role of data spaces we assume that the modulus of continuity $\overline{\omega}(r)$ satisfies the condition

$$\int_0^R \overline{\omega}(r)\,\frac{dr}{r} \leq C_R\,, \tag{0.8}$$

for some constant C_R. Assumption (0.8) is equivalent to the inclusion $D_{\overline{\omega}}(\overline{\Omega}) \subset C_*(\overline{\Omega})$.
We put each oscillation function $\overline{\omega}(r)$ as above, in correspondence with a unique, related oscillation function

$$\widehat{\omega}(r) = \int_0^r \overline{\omega}(s)\,\frac{ds}{s}\,. \tag{0.9}$$

Hence, to a functional space $D_{\overline{\omega}}(\overline{\Omega})$ there corresponds a well defined functional space $D_{\widehat{\omega}}(\overline{\Omega})$.
We show that

$$f \in D_{\overline{\omega}}(\overline{\Omega}) \quad \Longrightarrow \quad \nabla^2 u \in D_{\widehat{\omega}}(\overline{\Omega})\,.$$

It is convenient to impose some additional assumptions on the data spaces $D_{\overline{\omega}}(\overline{\Omega})$. Some assumptions on $\overline{\omega}(r)$, required in the sequel, can be substantially weakened. However, explicit statements in this direction would not add particularly significant features, at the cost of more involved manipulations.

We start by imposing the *strict* limitation

$$Lip(\overline{\Omega}) \subset D_{\overline{\omega}}(\overline{\Omega}) \subset C_*(\overline{\Omega}) . \tag{0.10}$$

Exclusion of $Lip(\overline{\Omega})$ means that $\overline{\omega}(r)$ does not verify $\overline{\omega}(r) \leq c\,r$, for any positive constant c. Hence $\limsup(\overline{\omega}(r)/r) = +\infty$, as $r \to 0$. We simplify, by assuming that

$$\lim_{r\to 0} \frac{\overline{\omega}(r)}{r} = +\infty . \tag{0.11}$$

In particular, the graph of $\overline{\omega}(r)$ is *tangent* to the vertical axis at the origin. This shows that *concavity* is here quite natural assumption. Concavity together with continuity and monotony shows that *point-wise differentiability*, for $r > 0$, is also a natural assumption.

Being $\overline{\omega}(r)$ concave, not flat, and differentiable, it follows that

$$\frac{\overline{\omega}(r)}{r\,\overline{\omega}'(r)} > 1\,, \tag{0.12}$$

for all $r > 0$. This justifies the assumption

$$\lim_{r \to 0} \frac{\overline{\omega}(r)}{r\,\overline{\omega}'(r)} = C_1 > 1\,, \tag{0.13}$$

where $C_1 = +\infty$ is admissible. This assumption is reenforced by the particular situation in Lipschitz, Hölder, and Log cases, where the limit is given by, respectively, 1, $\frac{1}{\lambda}$, and $+\infty$. As expected, the Lipschitz case stays outside the admissible range. Note that, basically, the larger is the space, the larger is the limit.

The above consideration allow us to assume that oscillation functions $\overline{\omega}(r)$, are concave, differentiable, and satisfies conditions (0.8), (0.11), and (0.13). Note that, due to a possible loss of regularity, it could happen that a $D_{\widehat{\omega}}(\overline{\Omega})$ space is not contained in $C_*(\overline{\Omega})$.

Theorem

Assume that the oscillation function $\overline{\omega}(r)$ *is concave and differentiable, and satisfies conditions* (0.8), (0.11), *and* (0.13). *Let* $\Omega_0 \subset\subset \Omega$, $f \in D_{\overline{\omega}}(\overline{\Omega})$, *and* u *be the solution of problem* (0.2), *where the differential operator* $\mathbf{L}$ *has constant coefficients. Then* $\nabla^2 u \in D_{\widehat{\omega}}(\Omega_0)$ *and*

$$\| \nabla^2 u \|_{\widehat{\omega}, \Omega_0} \leq C \| f \|_{\overline{\omega}} , \tag{0.14}$$

for some positive constant $C = C(\Omega_0, \Omega)$. *The above regularity holds up to flat boundary points. Furthermore, the result is optimal in the sharp sense defined below.*

Definition

We say that a given regularity statement of type $\overline{\omega} \to \widehat{\omega}$ is *sharp* if any regularity statement $\overline{\omega} \to \widehat{\omega}_0$, obtained by replacing $\widehat{\omega}$ by any other $\widehat{\omega}_0$, implies the existence of a constant c for which $\widehat{\omega}(r) \leq c\, \widehat{\omega}_0(r)$.

If the constant C_1 (see equation (0.13)) is finite then, in the previous theorem, full regularity occurs:

$$D_{\widehat{\omega}}(\overline{\Omega}) = D_{\overline{\omega}}(\overline{\Omega}) . \tag{0.15}$$

Furthermore, constant coefficients are not required. One has the following result.

Theorem

Assume that, in addition to the conditions required in theorem 5, the oscillation function $\overline{\omega}(r)$ *satisfies* (0.13) *for some* $C_1 < +\infty$. *Then* $\nabla^2 u \in D_{\overline{\omega}}(\overline{\Omega})$ *and*

$$\| \nabla^2 u \|_{\overline{\omega}} \leq C \| f \|_{\overline{\omega}} . \tag{0.16}$$

Note that regularity in the sharp sense immediately follows from full regularity.

The Log spaces $D^{0,\alpha}(\overline{\Omega})$. An intermediate regularity result.

Next we consider the particular case $\overline{\omega}(r) = \omega_\alpha(r)$, for any real $\alpha > 0$, where

$$\omega_\alpha(r) = (-\log r)^{-\alpha}\,, \tag{0.17}$$

$0 < r < 1$. So, the semi-norm $[\,f\,]_\alpha$ is the smallest constant for which

$$|f(x) - f(y)| \leq [\,f\,]_\alpha \cdot \left(\log \frac{1}{|\,x - y|} \right)^{-\alpha}. \tag{0.18}$$

Note that we have merely replaced, in the definition of the Hölder space $C^{0,\alpha}(\overline{\Omega})$, the quantity

$$\frac{1}{|\,x - y|} \quad \text{by} \quad \log \frac{1}{|\,x - y|}\,,$$

and allow α to be arbitrarily large.
As above, we define Banach spaces

$$D^{0,\alpha}(\overline{\Omega}) \equiv \{\, f \in C(\overline{\Omega}) : [\,f\,]_\alpha < \infty \,\}\,,$$

normalized by $\|\,f\,\|_\alpha = [\,f\,]_\alpha + \|\,f\,\|$.

One has

$$D^{0,\,\alpha}(\overline{\Omega}) \subset \; C_*(\overline{\Omega})$$

if and only if $\alpha > 1$.
Furthermore, for $0 < \beta < \alpha$, and $0 < \lambda \leq 1$,

$$C^{0,\,\lambda}(\overline{\Omega}) \subset D^{0,\,\alpha}(\overline{\Omega}) \subset D^{0,\,\beta}(\overline{\Omega}) \,\subset\, C(\overline{\Omega})\,. \tag{0.19}$$

Theorem

Let $\Omega_0 \subset\subset \Omega$, $f \in D^{0,\,\alpha}(\overline{\Omega})$ *for some* $\alpha > 1$, *and* u *be the solution of problem* (0.2), *where* $\mathbf{L}$ *has constant coefficients. Then* $\nabla^2 u \in D^{0,\,\alpha-1}(\Omega_0)$, *moreover*

$$\|\,\nabla^2 u\,\|_{\,\alpha-1,\,\Omega_0} \leq \; C\,\|\,f\,\|_{\alpha}\,, \tag{0.20}$$

for some positive constant $C = C(\alpha,\,\Omega_0,\,\Omega)$. *The regularity result holds up to flat boundary points. Results are optimal in the sharp sense.*

Sharp optimality is not confined to the particular family of spaces under consideration, but is something stronger. Let us illustrate the distinction.

Set $\omega(r) = \omega_{\nabla^2 u}(r)$. Theorem 8 claims that

$$\omega(r) \leq C_f \, (-\log r)^{-(\alpha-1)} \,, \tag{0.21}$$

for each $f \in D^{0,\alpha}(\overline{\Omega})$. Optimality of this result, *restricted* to the Log spaces' family, means that

$$\omega(r) \leq C_f \, (-\log r)^{-\beta} \tag{0.22}$$

is false for $\beta > \alpha - 1$. This does not exclude that (for instance) for all $f \in D^{0,\alpha}(\overline{\Omega})$ the oscillation $\omega(r)$ of $\nabla^2 u$ satisfies the estimate

$$\omega(r) \leq C_f \left[\log \left(\log \frac{1}{r} \right) \right]^{-1} \cdot (-\log r)^{-(\alpha-1)} \,,$$

which is weaker than (0.22), but stronger than (0.21).

Hölog spaces $C^{0,\lambda}_{\alpha}(\overline{\Omega})$ and full regularity.

If $\lambda\,\widehat{\omega}(r) = \overline{\omega}(r)$, for some $\lambda > 0$, then $\overline{\omega}(r) = k\,r^{\lambda}$, for some $k > 0$. This fact could suggest that Hölder spaces could be the unique full regularity class inside our framework. However, *full regularity* is also enjoyed by other spaces. The following is a particularly interesting case. Consider oscillation functions

$$\omega_{\lambda,\,\alpha}(r) = \left(\log\frac{1}{r}\right)^{-\alpha} r^{\lambda}\,, \qquad r < 1\,, \tag{0.23}$$

where $0 \leq \lambda < 1$ and $\alpha \in \mathbb{R}$, and define the B-spaces $C^{0,\lambda}_{\alpha}(\overline{\Omega})$.
For $\lambda = 0$ and $\alpha > 0$ we re-obtain the Log space $D^{0,\alpha}(\overline{\Omega})$.
For $\lambda > 0$ and $\alpha = 0$ we re-obtain the Hölder space $C^{0,\lambda}(\overline{\Omega})$.
On the other hand,

$$C^{0,\lambda}_{\alpha}(\overline{\Omega}) \subset C^{0,\lambda}_{\beta}(\overline{\Omega})\,, \quad \text{for} \quad \alpha > \beta > 0\,,$$

and

$$C^{0,\lambda_2}(\overline{\Omega}) \subset C^{0,\lambda}_{\alpha}(\overline{\Omega}) \subset C^{0,\lambda}(\overline{\Omega}) \subset C^{0,\lambda}_{-\alpha}(\overline{\Omega}) \subset C^{0,\lambda_1}(\overline{\Omega})\,,$$

for $0 < \lambda_1 < \lambda < \lambda_2 < 1$, and $\alpha > 0$.
All inclusions are compact.

The set

$$\bigcup_{\lambda,\,\alpha} C^{0,\,\lambda}_{\alpha}(\overline{\Omega}),$$

is a *totally ordered set*, in the set's inclusion sense. Roughly speaking, in the chain merely consisting of classical Hőlder spaces, each $C^{0,\,\lambda}$ space can be replaced by the infinite chain $C^{0,\,\lambda}_{\alpha}$, $\alpha \in \mathbb{R}$. The resulting chain is still totally ordered.

Theorem

Let $f \in C^{0,\,\lambda}_{\alpha}(\overline{\Omega})$ *for some* $\lambda \in (0,\, 1)$ *and some* $\alpha \in \mathbb{R}$. *Let* u *be the solution of problem* (0.2), *where the differential operator* $\boldsymbol{L}$ *may have variable coefficients. Then* $\nabla^2 u \in C^{0,\,\lambda}_{\alpha}(\overline{\Omega})$. *Moreover, for some* $C > 0$,

$$\|\,\nabla^2 u\,\|_{\lambda,\,\alpha} \leq C\,\|\,f\,\|_{\lambda,\,\alpha}\,. \tag{0.24}$$

Note that full regularity $\omega_{\lambda,\,\alpha} \to \omega_{\lambda,\,\alpha}$ could be a little surprising here. In fact, one could merely expected the intermediate regularity result

$$\omega_{\lambda,\,\alpha} \to \omega_{\lambda,\,\alpha-1}\,.$$

The proof follows from theorem 5. Set $\omega_{\lambda,\,\alpha}(r) = \overline{\omega}(r)$. Assumptions (0.8) and (0.11) are trivially verified. Set $L(r) = \log \frac{1}{r}$. One shows that

$$\overline{\omega}'(r) = r^{\lambda-1}\, L(r)^{-\alpha}\, (\,\lambda + \alpha\, L(r)^{-1}\,)$$

and that

$$\overline{\omega}''(r) = -\, r^{\lambda-2}\, L(r)^{-\alpha} \left(\lambda\,(1-\lambda) - (\,2\lambda - 1)\,\alpha\, L(r)^{-1} - \alpha\,(\alpha + 1)\, L(r)^{-2}\right).$$

Concavity of $\overline{\omega}$ follows, since $\overline{\omega}''(r) < 0$ in a neighborhood of the origin. Furthermore (0.13) holds since

$$\lim_{r \to 0} \frac{\overline{\omega}(r)}{r\, \overline{\omega}'(r)} = \frac{1}{\lambda} = C_1 > 1\,. \tag{0.25}$$

Lastly, by de l'Hôpital rule, we get

$$\lim_{r \to 0} \frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r \to 0} \frac{\overline{\omega}(r)}{r\, \overline{\omega}'(r)}\,. \tag{0.26}$$

Hence, by appealing to (0.25), we show that the limit is positive and finite. Full regularity follows.