Homogenization of nonlinear shell models in elasticity

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Calculus of Variations and its Applications (on the occasion of Luísa Mascarenhas' 65th birthday)

December 17–19, 2015.

Outline

- Plate models in elasticity (von Kármán and bending);
- Homogenization and dimensional reduction;
- Review of the results for homogenization on von Kármán and bending plate and shells;

Plate models in elasticity

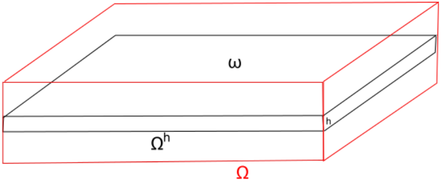
Minimization functional of $3D$ elasticity

$$\int_\Omega W(\nabla \boldsymbol{u})dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx,$$

- $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ deformation
- $\boldsymbol{f} : \Omega \to \mathbb{R}^3$, external volume dead loads
- $W : \mathbb{R}^{3\times 3} \to [0, \infty]$, stored energy function with the properties (important for higher order models)
    1. class C^2 in a neighborhood of $\mathrm{SO}(3)$;
    2. W is frame-indifferent, i.e., $W(\mathbf{F}) = W(\mathbf{RF})$ for every $\mathbf{F} \in \mathbb{R}^{3\times 3}$ and $\mathbf{R} \in \mathrm{SO}(3) \Leftrightarrow W(\mathbf{F}) = W(\sqrt{\mathbf{F}^T\mathbf{F}})$;
    3. $W(\mathbf{F}) \geq C_W \operatorname{dist}^2(\mathbf{F}, \mathrm{SO}(3))$, for some $C_W > 0$ and all $\mathbf{F} \in \mathbb{R}^{3\times 3}$, $W(\mathbf{F}) = 0$ iff $\mathbf{F} \in \mathrm{SO}(3)$.

Plate models in elasticity

- $$\int_{\Omega^h} W(\nabla \boldsymbol{u}^h)dx - \int_{\Omega^h} \boldsymbol{f}^h \cdot \boldsymbol{u}^h dx,$$
  $\Omega^h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, $\omega \subset \mathbb{R}^2$ Lipschitz domain.

  What happens as $h \to 0$?
  Rescaling: $P^h : \Omega \to \Omega^h$, $P^h(x_1, x_2, x_3) = (x_1, x_2, hx_3)$,
  $\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. Minimization functional:

  $$\int_{\Omega} W(\nabla_h \boldsymbol{u}^h)dx - \int_{\Omega} \boldsymbol{f}^h \cdot \boldsymbol{u}^h dx,$$

  $\nabla_h = \nabla_{\boldsymbol{e}_1, \boldsymbol{e}_2} + \frac{1}{h}\nabla_{\boldsymbol{e}_3}$-physical gradient translated on the canonical domain.

Plate models in elasticity-hierarchy of models

Assumption: free boundary condition

$$\frac{1}{h^\alpha}\int_\Omega W(\nabla_h \boldsymbol{u})dx \xrightarrow{\Gamma} ?$$

- $\alpha = 0$ membrane model;
- $0 < \alpha < 5/3$ trivial limit; zero for short map ($(\nabla \boldsymbol{u})^T \nabla \boldsymbol{u} \leq \mathbf{I}$); infinity for the others;
- $5/3 \leq \alpha < 2$ open (short maps have non-zero energy?);
- $\alpha = 2$ bending model;
- $2 < \alpha < 4$ constrained models;
- $\alpha = 4$ von Kármán model;
- $\alpha > 4$ linear von Kármán theory

Clamped plate: $0 < \alpha < 4$ Föppl displacement theory.
Rod: due to simpler geometry in regime $0 < \alpha < 2$ trivial.

Homogenization and dimensional reduction-membrane case of plate

Assumption: $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0, \infty\rangle$.

$\omega \subset \mathbb{R}^2$ domain; $\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$; $\Omega^h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$

Theorem (Braides, Fonseca, Francfort)

Ω domain, $Y = [0,1]^2$, $\mathcal{Y} = [0,1\rangle^2$-torrus, $W : \mathcal{Y} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ continuous with p growth and coercivity assumption

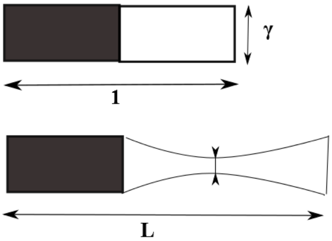
$$I^h(\boldsymbol{u}) = \int_\Omega W(\tfrac{x_\alpha}{\varepsilon(h)}, \nabla_h \boldsymbol{u})dx \xrightarrow[W^{1,p}]{\Gamma} I(\boldsymbol{u}) = \int_\omega W_{hom}(\nabla \boldsymbol{u})d\hat{x};$$

$\mathcal{S}_k = \{\psi \in W^{1,p}(kY \times I; \mathbb{R}^3) : \psi = 0 \text{ on lateral boundary }\}$. *The cell formula is given by:*

$$W_{hom}(\mathbf{G}) = \inf_{k\in\mathbb{N}} \inf_{\psi\in\mathcal{S}_k} \left\{ \frac{1}{k^2} \iint_{kY\times I} W\Big(y, \mathbf{G} + \big(\nabla_y \psi | \tfrac{1}{\gamma}\partial_{x_3}\psi\big)\Big) dydx_3 \right\}$$

Homogenization and dimensional reduction-membrane case of plate

Relaxation: $\mathbf{G}\hat{x} + \varepsilon(h)\psi(\hat{x}, \frac{\hat{x}}{k\varepsilon(h)}, \frac{x_3}{\gamma})$, e.g. $k = 1$



Homogenization and dimensional reduction-convex case with two-scale convergence

$\boldsymbol{u} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, $\Omega = \omega \times I$, $I = [-\frac{1}{2}, \frac{1}{2}]$ rescaled domain, $W : \mathcal{Y} \times \mathbb{M}^3 \to \mathbb{R}$, convex with p-growth and coercivity assumption $\mathcal{Y} = [0,1)^2$ with the topology of torrus.
The rescaled and renormalized energy functional

$$\int_\Omega W\left(\frac{x'}{\varepsilon(h)}, \nabla_h \boldsymbol{u}\right)$$

$x' = (x_1, x_2)$.
Compactness question: How do the two scale limits of the rescaled gradients look like, if we know that $\limsup_{h\to 0} \|\nabla_h \boldsymbol{u}^h\|_{W^{1,p}} < \infty$? Basic heuristics:

$$u^h(x_1, x_2, x_3) = u_0(x_1, x_2) + h u_1\left(x', \frac{x'}{\varepsilon(h)}, x_3\right)?$$

Homogenization and dimensional reduction-two scale limits of scaled gradient

Let $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma$. Claim (Neukamm, Phd thesis '10):

1. $h \sim \varepsilon(h)$ i.e. $\gamma \in \langle 0, \infty\rangle$:
   $u^h = u_0(x') + hu_1(x', \frac{x'}{\varepsilon(h)}, x_3)$
   $\nabla_h \boldsymbol{u}^h \xrightarrow{2} (\nabla'_x \boldsymbol{u}_0 \,|\, 0) + (\nabla_y \boldsymbol{u}_1, \frac{1}{\gamma}\partial_{x_3} u_1)$,
2. $h \ll \varepsilon(h)$ i.e. $\gamma = 0$,
   $\boldsymbol{u}^h = u_0(x') + \varepsilon(h)\boldsymbol{u}_1(x', \frac{x'}{\varepsilon(h)}) + h\boldsymbol{u}_2(x', \frac{x'}{\varepsilon(h)}, x_3)$,
   $\nabla_h \boldsymbol{u}^h \xrightarrow{2} (\nabla'_x \boldsymbol{u}_0 \,|\, 0) + (\nabla_y \boldsymbol{u}_1, \partial_{x_3} \boldsymbol{u}_2)$,
3. $h \gg \varepsilon(h)$ i.e. $\gamma = \infty$,
   $\boldsymbol{u}^h = \boldsymbol{u}_0(x') + \varepsilon(h)\boldsymbol{u}_1(x', \frac{x'}{\varepsilon(h)}, x_3) + h\boldsymbol{u}_2(x', x_3)$
   $\nabla_h \boldsymbol{u}^h \xrightarrow{2} (\nabla'_x \boldsymbol{u}_0 \,|\, 0) + (\nabla_y \boldsymbol{u}_1, \partial_{x_3} \boldsymbol{u}_2)$.

Remark: The energy is obtained by relaxing with the appropriate corrector. Different degrees of freedom in the corrector imply different models.

Homogenization and dimensional reduction

Remark: The analysis of the von Kárman plate is done by Neukamm & V (M3AS '13). The situation when $h \ll \varepsilon(h)$ corresponds to the case when dimensional reduction prevails and the obtained model is the homogenized von Kármán plate model. The situation when $\varepsilon(h) \ll h$ corresponds to the situation when homogenization prevails and the obtained model is the same as the von Kármán model for the homogenized material. Both of these energies can be obtained as limit cases of the energy $\gamma \to 0$ i.e. $\gamma \to \infty$. The same phenomenology is for the bending case of rod (Neukamm, ARMA '12), while not for von Kármán case of shell (Hornung & V., AHP '14) nor for bending case of plate (Hornung, Neukamm, V. CV '14; V. CV '15).

Simultaneous homogenization and dimensional reduction-vK plate

Assumption: $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0, \infty \rangle$.

$$\boldsymbol{u}^h_{\mathrm{vK}} = \underbrace{\begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{u} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix}}_{\text{compactness}}$$

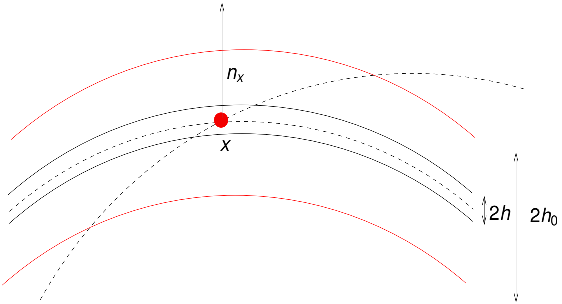
$$+ \underbrace{h^3 x_3 \boldsymbol{d}_1 + h^3 x_3^2 \boldsymbol{d}_2}_{\text{relaxation}}$$

$$\boldsymbol{u}^h_{\mathrm{HvK}} = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{u} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix}$$

$$+ h^2 \varepsilon(h) \psi\left(x_1, x_2, \frac{x_1}{\varepsilon(h)}, \frac{x_2}{\varepsilon(h)}, x_3\right)$$

Simultaneous homogenization and dimensional reduction-vK shell

Recall ordinary von Kármán shell:



$$S^h = \{z = x + t\boldsymbol{n}(x); x \in S, -h/2 < t < h/2\},$$

$$\boldsymbol{\Pi}(x) = \nabla \boldsymbol{n}(x) \text{ shape operator }.$$

Rescale the problem on the domain S^1 and define appropriate $\nabla_h \boldsymbol{u}^h$.

Simultaneous homogenization and dimensional reduction-vK shell

The typical deformation of order 4 looks like

$$\boldsymbol{u}^h(x + t\boldsymbol{n}(x)) = x + h\boldsymbol{v} + th^2\mathbf{A}\boldsymbol{n} + h^2\boldsymbol{w}^h + \textit{relaxation},$$

where $\boldsymbol{v} \in H^1(S;\mathbb{R}^3)$ is an infinitezimal isometry i.e.

$$\partial_{\boldsymbol{\tau}} v = \mathbf{A}(x)\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in T_x S \text{ a.e. } x \in S,$$

and

$$\mathbf{A} \in H^1(S;\mathbb{R}^{3\times 3}),\ \mathbf{A} \text{ antisymmetric } \forall x \in S.$$

However it is not necessary neither the compactness result provide us that $\boldsymbol{w}^h$ is bounded in $H^1(S;\mathbb{R}^3)$. Only what compactness result guarantees is that

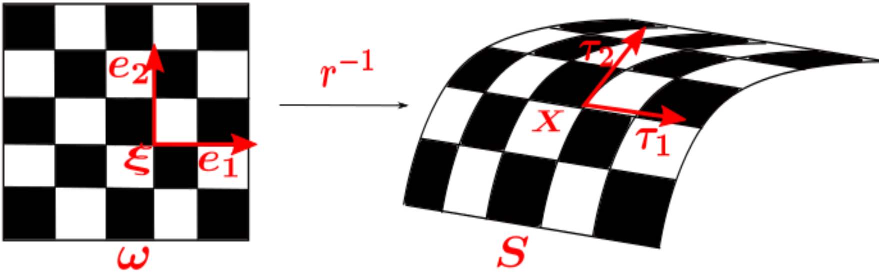
$$\mathrm{sym}\,\nabla \boldsymbol{w}^h = \mathrm{sym}\,\nabla \boldsymbol{w}^h_{\tan} + (\boldsymbol{w}^h\boldsymbol{n})\boldsymbol{\Pi},\ \text{where } \boldsymbol{w}^h = \boldsymbol{w}^h_{\tan} + \langle \boldsymbol{w}^h, \boldsymbol{n}\rangle \boldsymbol{n},$$

is bounded in $L^2(\omega;\mathbb{M}^2_{\mathrm{sym}}(T_x S))$. This does not guarantee boundedness of w^h even in L^2 or H^{-1}.

Simultaneous homogenization and dimensional reduction-vK shell

Assumption: global parametrization of class C^3, $\boldsymbol{r} : S \to \bar{\omega}$.

$$\boldsymbol{\tau}_1(x) = \partial_{\xi_1} \boldsymbol{r}^{-1}|_{\boldsymbol{r}(x)},\ \boldsymbol{\tau}_2(x) = \partial_{\xi_2} \boldsymbol{r}^{-1}|_{\boldsymbol{r}(x)}.$$



Simultaneous homogenization and dimensional reduction-vK shell

Theorem (Hornung,V.)

The cell formula for homogenized von Kármán shell, under the assumption that $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0,\infty\rangle$ *is given by*

$$Q_\gamma(x,\mathbf{M}^1_{tan},\mathbf{M}^2_{tan}) = \inf_{\psi\in H^1(I\times\mathcal{Y};\mathbb{R}^3)} \iint_{I\times\mathcal{Y}} Q\Big(y,\Lambda(\mathbf{M}^1_{tan},\mathbf{M}^2_{tan})$$

$$+\sum_{i,j=1,2,3} \mathrm{sym}\left(\nabla_y\psi(t,y),\tfrac{1}{\gamma}\partial_t\psi(t,y)\right)_{ij}\boldsymbol{\tau}^i(x)\otimes\boldsymbol{\tau}^j(x)\Big)dtdy,$$

Here $Q(y,\cdot) = \frac{\partial^2 W}{\partial\mathbf{F}^2}(y,\mathbf{I})$ *is quadratic form,* $I=[-\frac{1}{2},\frac{1}{2}]$ *and* $\Lambda(\mathbf{M}^1_{tan},\mathbf{M}^2_{tan}) = \mathbf{M}^1_{tan}+t\mathbf{M}^2_{tan}$. *The limit energy functional is given by*

$$I_{hom}(\boldsymbol{v},\mathbf{B}_{tan}) = \int_S Q_\gamma\Big(x,\mathbf{B}_{tan}-\tfrac{1}{2}(\mathbf{A}^2)_{tan},\left(\mathrm{sym}\,\nabla(\mathbf{A}\boldsymbol{n})-\mathrm{sym}(\mathbf{A}\boldsymbol{\Pi})\right)_{tan}\Big)dx.$$

Simultaneous homogenization and dimensional reduction-vK shell

- the cases $\varepsilon(h) \ll h$ and $h \sim \varepsilon(h)$ are similar to the plate case (add the appropriate corrector).
- The case $\gamma = \infty$ corresponds to the situation where homogenization dominates and is the limit case when $\gamma \to \infty$.
- The case $\gamma = 0$ is more subtle and causes further hierarchy of models; we are able to obtain the models for $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ and $h \sim \varepsilon(h)^2$ for the generic shell;
- in the case $h \ll \varepsilon(h)^2$ we are able to obtain the model for convex shell

Simultaneous homogenization and dimensional reduction-bending plate and shell

1. in the case of bending plate we are able to obtain the models for $\varepsilon(h) \ll h$, $h \sim \varepsilon(h)$ and $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$;
2. in the case $h \sim \varepsilon(h)^2$ it could be that oscillations that are not on the level of the oscillations of material appear in the relaxation.
3. the homogenization of $2d$ plate model, under the constraint of being isometry is analyzed by Neukamm and Olbermann (CV'15).
4. in the case of bending shell the limit model heavily depends on the geometry of shell. For the simplest case of convex shell we have the partial result; it can be seen that we have the same phenomenology as in the case of bending plate, but we can construct recovery sequence only for regular enough isometries (we are able to prove that $W^{2,2}$ isometries of convex shell have C^∞ inner regularity, but we need additional boundary regularity).

Simultaneous homogenization and dimensional reduction-remarks

- in the case of von Kárman plate and bending rod we are able to obtain the models without periodicity assumption; (V. AA 2015; Marohnić, V. AMPA 2015.)
- we are able to prove the locality of Γ- closure (joint work with M. Bukal).

Thank you for your attention!