Hybrid analytical and integral methods for simulating HTS materials

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Why analytical modeling?

- In the design phase, any analytical solution with 80% accuracy is very valuable.

- Of course, there are still difficulties in numerical modeling of large scale systems including HTS.

- A reference solution is still needed.

- We should continue to develop such kind of models.
What problems can be solved?

- 2D problems with appropriate boundary conditions
Pulsed Field Magnetization

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Outline

- Problem description (analytical method)
  - Formulation and assumptions
  - Separation of variables method
  - Inductance calculation

- Brandt’s method (integral method)
  - Results of trapped field and induced currents
  - Computing times

- Future work (and collaborations)
Inductor with magnetic circuit including an airgap

- The coil is fed with a uniform current density $J$
- Dirichlet and Neuman boundary conditions
- Boundaries are far enough $z_1 \gg 0$ and $z_5 \gg z_4$
- HTS bulk will be later placed in the airgap
Formulation and assumptions

- Magnetic vector potential in cylindrical coordinates

- The problem is axisymmetric so \( \mathbf{A} = A_\theta (r, z) \mathbf{u}_\theta \)

- All materials are assumed magnetically linear
  - Superposition theorem can apply

- From Maxwell’s equations with Coulomb gauge

\[
\begin{align*}
\nabla^2 A_\| &= -\mu_0 \mathbf{J} \quad \text{for Region II (coil)} \\
\nabla^2 A_i &= 0 \quad \text{for Region } i = \text{I,II and IV}
\end{align*}
\]
Separation of variables method
from PDE to ODE

- Assuming that the magnetic vector potential is \( A(r, z) = R(r)Z(z) \)

\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = 0 \iff \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - \frac{1}{r^2} = -\frac{Z''}{Z} = \alpha^2
\]

\[
\Rightarrow \begin{cases} 
Z'' + \alpha^2 Z = 0 \\
r^2 R'' + rR' - (1 + \alpha^2 r^2)R = 0 \quad \text{(modified Bessel equation)}
\end{cases}
\]

- Each solution must verify the ODE in the considered Region as well as the Boundary Conditions
In Region II (coil)

\[ \frac{\partial^2 A_{\|}}{\partial^2 r} + \frac{1}{r} \frac{\partial A_{\|}}{\partial r} - \frac{A_{\|}}{r^2} + \frac{\partial^2 A_{\|}}{\partial^2 z} = -\mu_0 J(z) \]

1) Looking for the eigenvalues and the eigenfunctions of the homogeneous equation which satisfy the BC

\[ Z_{\|}(z) = b_{n,z}^{\|} \sin(\alpha_n z) \]  \text{with}  \ \alpha_n = \frac{n\pi}{Z_5}

2) The source term can be expand in terms of the eigenfunctions

\[ J(z) = \sum_{n=1}^{\infty} J_n \sin(\alpha_n z) \]  \text{with}  \ \alpha_n = \frac{2J}{n\pi} \left( \cos(\alpha_n Z_1) - \cos(\alpha_n Z_4) \right)
3) The general solution requires the solution of a non-homogeneous Bessel’s differential equation on the $r$-variable

$$A_{II}(r, z) = \sum_{n=1}^{\infty} \left( a_n^{II} I_1(\alpha_n r) + b_n^{II} K_1(\alpha_n r) - C_n L_1(\alpha_n r) \right) \sin(\alpha_n z)$$

with $C_n = \frac{\mu_0 \pi J_n}{2 \alpha_n^2}$

- $I_1$ and $K_1$ are the modified Bessel functions of the first and second kind and order 1, $L_1$ is the modified Struve function of order 1
- The integration constants $a_n^{II}, b_n^{II}$ have to be determined from interface conditions
In all Regions

\[ A_I(r, z) = \sum_{n=1}^{\infty} b'_n \left( \frac{K_0(\alpha_n R_4)}{I_0(\alpha_n R_4)} l_1(\alpha_n r) + K_1(\alpha_n r) \right) \sin(\alpha_n z) \] with \( \alpha_n = n\pi / Z_5 \)

\[ A_{II}(r, z) = \sum_{n=1}^{\infty} \left( a''_n l_1(\alpha_n r) + b''_n K_1(\alpha_n r) - C_n l_1(\alpha_n r) \right) \sin(\alpha_n z) \]

\[ A_{III}(r, z) = \sum_{n=1}^{\infty} \left( a'''_n l_1(\alpha_n r) + b'''_n K_1(\alpha_n r) \right) \sin(\alpha_n z) \]

\[ A_{IV}(r, z) = A_0 r + \sum_{k=1}^{\infty} a''_k l_1(\beta_k r) \cos(\beta_k (z - Z_2)) \] with \( \beta_k = k\pi / (Z_3 - Z_2) \)

with \[ A_0 = \left( R_1 (Z_3 - Z_2) \right)^{-1} \sum_{n=1}^{\infty} \alpha^{-1}_n \left( \cos(\alpha_n Z_2) - \cos(\alpha_n Z_3) \right) \left( a'''_n l_1(\alpha_n R_1) + b'''_n K_1(\alpha_n R_1) \right) \]
Interface conditions

- At \( r = R_3 \)

\[
\begin{align*}
A_I(r = R_3, z) &= A_{II}(r = R_3, z) \\
\partial_r (r A_I(r, z))_{r=R_3} &= \partial_r (r A_{II}(r, z))_{r=R_3}
\end{align*}
\]

\( \iff \) continuity of \( A \)

\( \iff \) tangential continuity of \( B \)

\[
\begin{bmatrix}
\eta_{11} & \eta_{12} & \eta_{13} & 0 & 0 & 0 \\
\eta_{21} & \eta_{22} & \eta_{23} & 0 & 0 & 0 \\
0 & \eta_{32} & \eta_{33} & \eta_{34} & \eta_{35} & 0 \\
0 & \eta_{42} & \eta_{43} & \eta_{44} & \eta_{45} & 0 \\
0 & 0 & 0 & \eta_{54} & \eta_{55} & \eta_{56} \\
0 & 0 & 0 & \eta_{64} & \eta_{65} & \eta_{66}
\end{bmatrix}
\begin{bmatrix}
b^I_n \\
a^II_n \\
b^II_n \\
a^III_n \\
b^III_n \\
a^IV_n
\end{bmatrix}
\begin{bmatrix}
-C_n L_1 (\alpha_n R_3) \\
-C_n L_0 (\alpha_n R_3) \\
-C_n L_1 (\alpha_n R_2) \\
-C_n L_0 (\alpha_n R_2) \\
0 \\
0
\end{bmatrix}
\]

\[
\eta_{11} = \frac{K_0 (\alpha_n R_4)}{I_0 (\alpha_n R_4)} I_1 (\alpha_n R_3) + K_1 (\alpha_n R_3), \eta_{12} = -I_1 (\alpha_n R_3), \eta_{13} = -K_1 (\alpha_n R_3)...
\]
Inductance calculation

\[ \phi = \frac{2\pi}{S} \int_{Z_1}^{Z_4} \left( \int_{R_3}^{R_2} A_{IL} r dr \right) dz, \quad \Phi = N\phi = L_{mc} I \text{ and } NI = JS \]

\[ L_{mc} = 2\pi \frac{N^2}{JS^2} \sum_{n=1}^{N_{\text{max}}} \left\{ \frac{\pi}{2\alpha_n^2} \left( a_n'' \left( R_3 U(R_3) - R_2 U(R_2) \right) + b_n'' \left( R_3 V(R_3) - R_2 V(R_2) \right) \right) \right. \\
\left. - \mu_0 J_n \frac{\alpha_n}{6\pi} \left( R_3^4 W(R_3) - R_2^4 W(R_2) \right) \right\} \left( \cos(\alpha_n Z_1) - \cos(\alpha_n Z_4) \right) \]

\[ U(r) = I_1(\alpha_n r) L_0(\alpha_n r) - I_0(\alpha_n r) L_1(\alpha_n r), \]
\[ V(r) = K_1(\alpha_n r) L_0(\alpha_n r) + K_0(\alpha_n r) L_1(\alpha_n r), \]
\[ W(r) = F \left\{ \left\{1, 2 \right\}; \left\{ \frac{3}{2}, \frac{5}{2}, 3 \right\}; \frac{\alpha_n^2 r^2}{4} \right\} \]

\[ \varepsilon < 3\% \quad \text{with } N_{\text{max}} \geq 20 \]
Coupling with Brandt’s method

The magnetic vector potential in the airgap is used as a source term.

The applied current density is now

- pulsed \( J_a(t) = J_{\text{max}} \left( \frac{t}{\tau} \right) \exp \left( 1 - \left( \frac{t}{\tau} \right) \right) \)

\[
\frac{dJ(r_2,t)}{dt} = -\mu_0^{-1} \int_{Z_{22}}^{Z_{33}} \int_{0}^{R_{11}} Q_{\text{cyl}}^{-1}(r_2, r'_2) \left( E(J) + \frac{dA_{iv}(r'_2, t)}{dt} \right) dr'dz' \text{ with } r_2 = (r, z)
\]
Coupling with Brandt’s method

**integral method**

\[
Q_{cyl}(r_2, r'_2) = \frac{1}{2\pi} \int_0^\pi \frac{r' \cos \varphi \, d\varphi}{(z - z')^2 + r^2 + r'^2 - 2rr' \cos \varphi}^{1/2}
\]

with \( \varphi = \arctan(z / r) \)

- \( Q_{cyl} \) is the kernel obtained by integrating the 3D Green function
- The ferromagnetic material influence is taken into account by means of the method of images
Results

Applied magnetic field $B_a$ and trapped magnetic field $B_{\text{trap}}$ on the top surface of the bulk HTS, at $r = 0$ and $z = Z_{33}$.

Normalized current density $J / J_c$ in the middle plane of the bulk HTS for $r = \{0, 4.7, 9.5, 15.3, 20\}$ mm.

$$E(J) = \left( \frac{E_c}{J_c} \right) \left( \frac{|J|}{J_c} \right)^{n-1} J \quad \text{with} \quad J_c = 100 \text{ A/mm}^2, \quad n = 30, \quad E_c = 1 \mu\text{V/cm}$$
Computing times

Intel® Core™ i7-4600 CPU @ 2.10 GHz (Turbo Boost 2.70 GHz), 12 GB RAM (my laptop)

- Analytical calculation
  - < 1 s
- Construction of the topological matrices
  - 75 s
- Solving the system using ode23t (Matlab)
  - 642 s

- COMSOL 5.2a (using PDE)
  - 320 s with linear elements
  - 1896 s with quadratic elements
Future work

- Improve matrix conditioning
  - filtering, cut-off...
- Symbolic expression of the coefficients
  - $a_n^1, b_n^1, a_n^2, b_n^2, a_n^3, b_n^3$ and $a_n^4$
- Coupling with thermal equation
  - using finite difference method
- Adaptive mesh refinement
  - with the current density edge
- 3D problems (multiple bulks, magnetic lens...)
  - e.g. with “3D Simulation using the Fast Fourier Transform” from Leonid Prigozhin ([https://arxiv.org/abs/1803.01346](https://arxiv.org/abs/1803.01346))
Thank you for your attention!

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