HTS 2018 MODELLING
6th International Workshop on Numerical Modelling of High Temperature Superconductors

Hybrid analytical and integral methods for simulating HTS materials

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Why analytical modeling?

- In the design phase, any analytical solution with 80% accuracy is very valuable

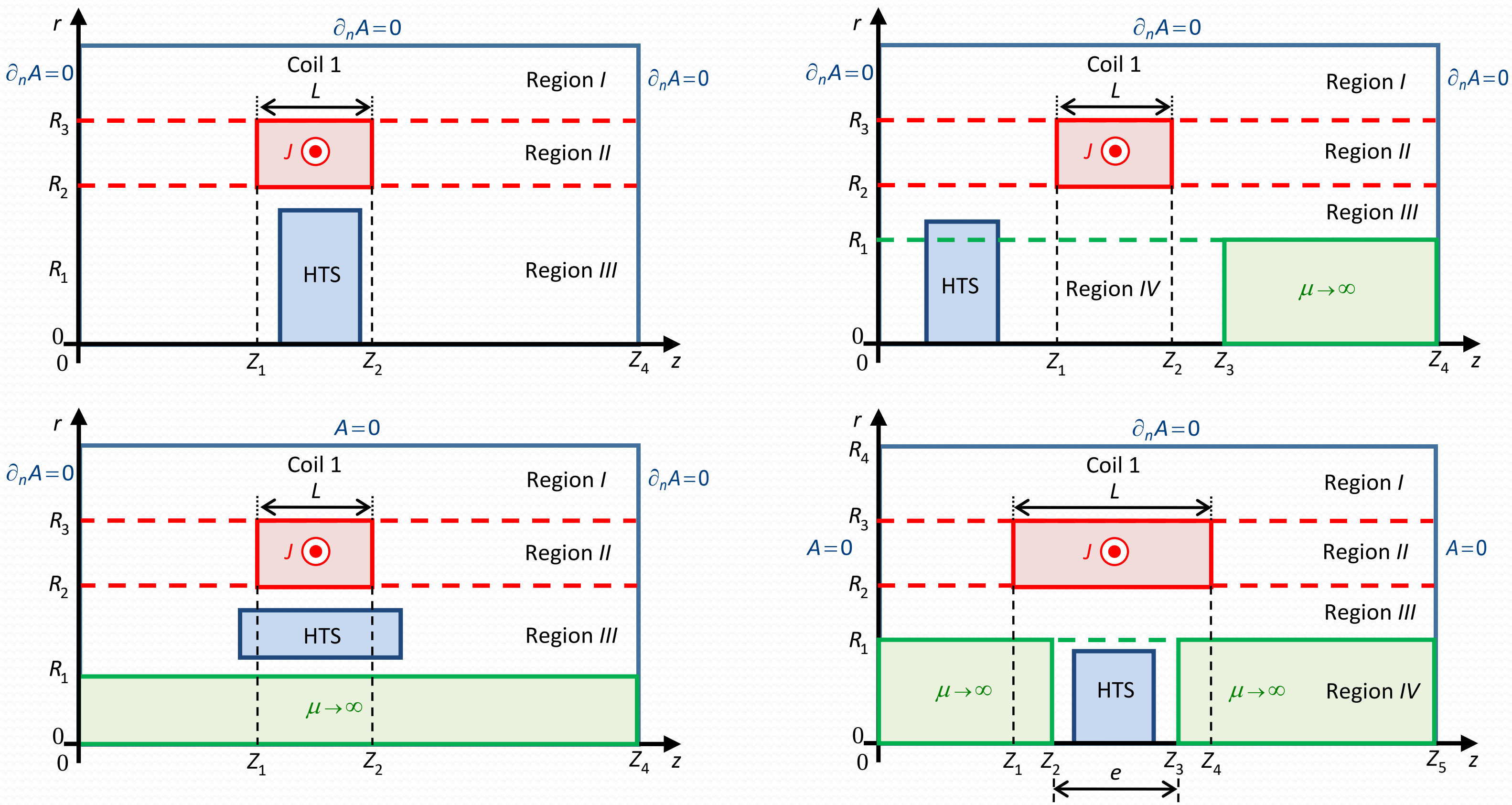
- Of course, there are still difficulties in numerical modeling of large scale systems including HTS

- A reference solution is still needed

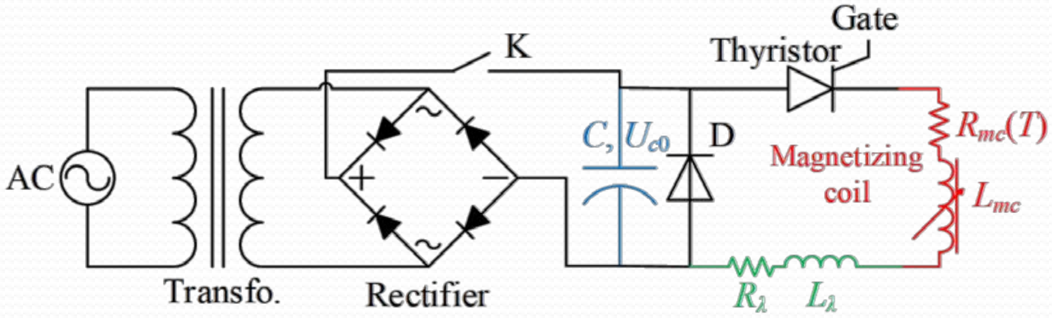
- We should continue to develop such kind of models

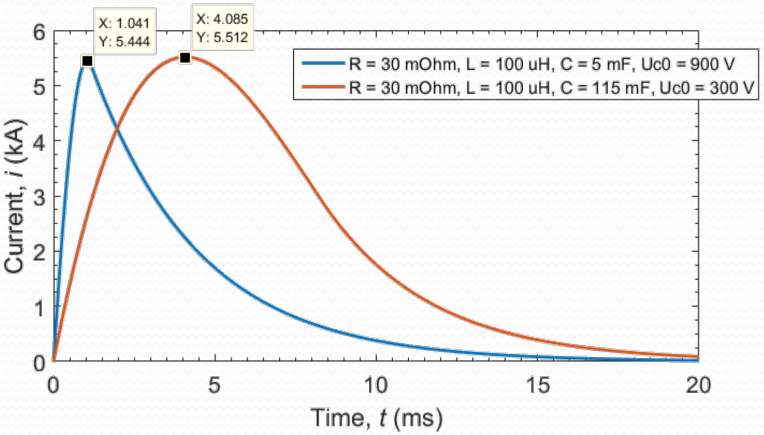
What problems can be solved?

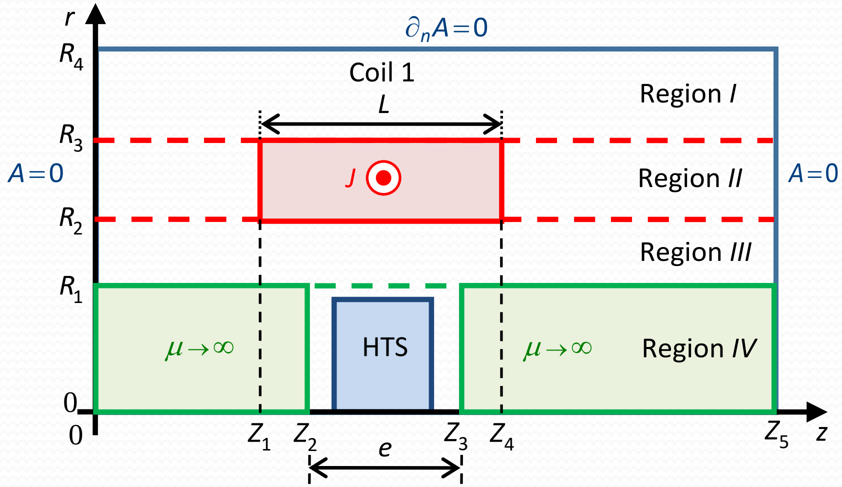
- 2D problems with appropriate boundary conditions

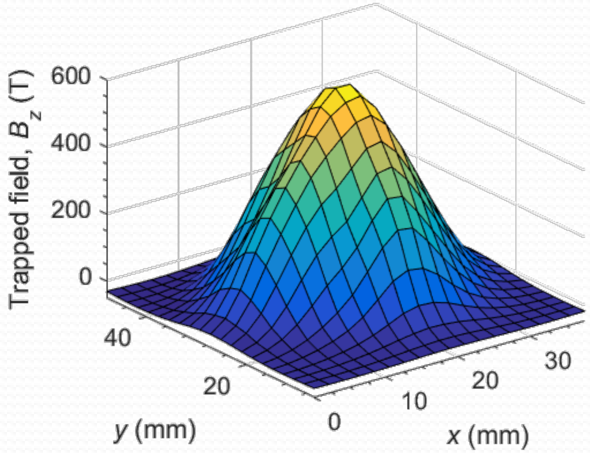


Pulsed Field Magnetization









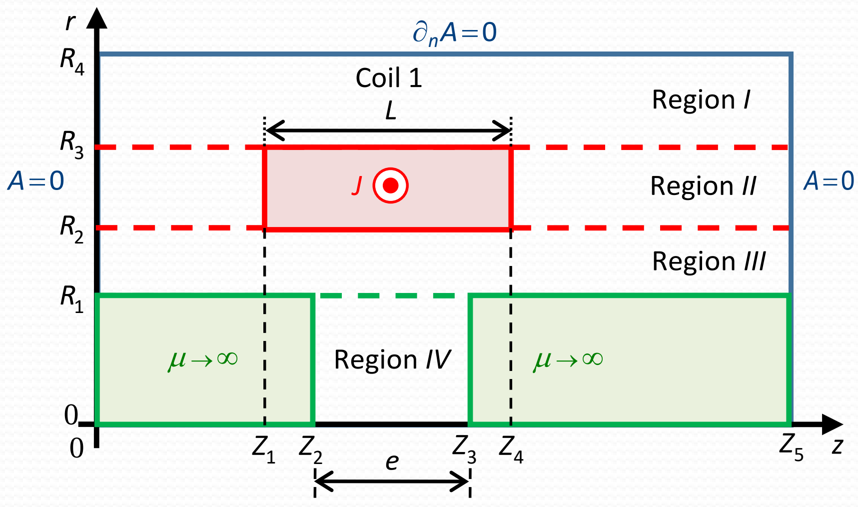
Outline

- Problem description (analytical method)
  - Formulation and assumptions
  - Separation of variables method
  - Inductance calculation
- Brandt's method (integral method)
  - Results of trapped field and induced currents
  - Computing times
- Future work (and collaborations)

Problem description

analytical method

Inductor with magnetic circuit including an airgap



- The coil if fed with a uniform current density *J*
- Dirichlet and Neuman boundary conditions
- Boundaries are far enough

  $z_1 >> 0$ and $z_5 >> z_4$
- HTS bulk will be later placed in the airgap

Formulation and assumptions

- Magnetic vector potential in cylindrical coordinates
- The problem is axisymmetric so $\mathbf{A} = A_\theta(r,z)\mathbf{u}_\theta$
- All materials are assumed magnetically linear
  - Superposition theorem can apply
- From Maxwell's equations with Coulomb gauge

$$\begin{cases} \nabla^2 \mathbf{A}_{\mathrm{II}} = -\mu_0 \mathbf{J} & \text{for Region II (coil)} \\ \nabla^2 \mathbf{A}_i = 0 & \text{for Region } i = \text{I,II and IV} \end{cases}$$

Separation of variables method

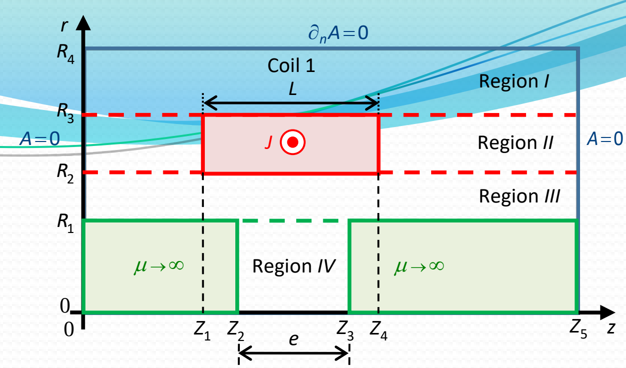
from PDE to ODE

- Assuming that the magnetic vector potential is $A(r,z) = R(r)Z(z)$

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = 0 \Leftrightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = -\frac{Z''}{Z} = \alpha^2$$

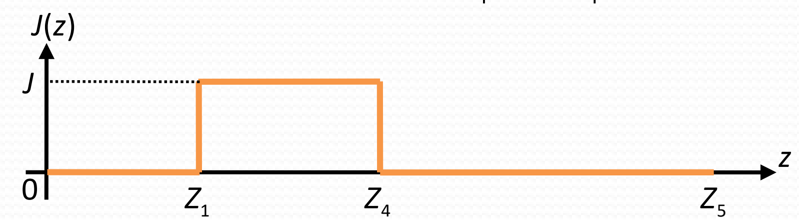
$$\Rightarrow \begin{cases} Z'' + \alpha^2 Z = 0 \\ r^2 R'' + rR' - (1 + \alpha^2 r^2)R = 0 \quad \text{(modified Bessel equation)} \end{cases}$$

- Each solution must verify the ODE in the considered Region as well as the Boundary Conditions



In Region II (coil)

$$\frac{\partial^2 A_{II}}{\partial^2 r}+\frac{1}{r}\frac{\partial A_{II}}{\partial r}-\frac{A_{II}}{r^2}+\frac{\partial^2 A_{II}}{\partial^2 z}=-\mu_0 J(z)$$



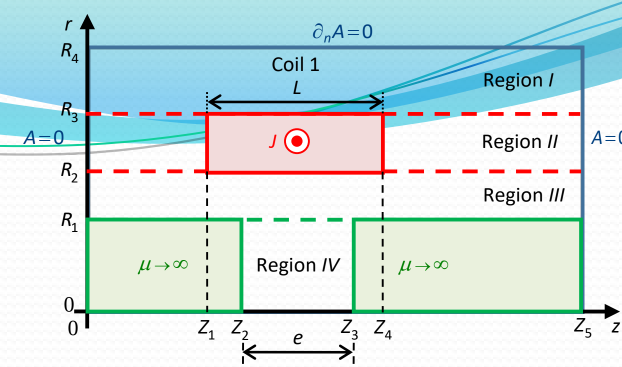
1) Looking for the eigenvalues and the eigenfunctions of the homogeneous equation which satisfy the BC

$$Z_{II}(z)=b^{II}_{n,z}\sin(\alpha_n z)\ \text{ with } \alpha_n=n\pi/Z_5$$

2) The source term can be expand in terms of the eigenfunctions

$$J(z)=\sum_{n=1}^{\infty} J_n \sin(\alpha_n z)\ \text{ with } J_n=\frac{2J}{n\pi}\left(\cos(\alpha_n Z_1)-\cos(\alpha_n Z_4)\right)$$

In Region II (coil)

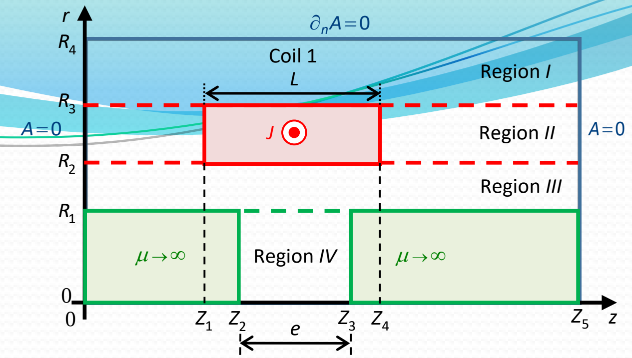


3) The general solution requires the solution of a non-homogeneous Bessel's differential equation on the r-variable

$$A_{II}(r,z)=\sum_{n=1}^{\infty}\left(a_n^{II} I_1(\alpha_n r)+b_n^{II} K_1(\alpha_n r)-C_n L_1(\alpha_n r)\right)\sin(\alpha_n z) \text{ with } C_n=\frac{\mu_0 \pi J_n}{2\alpha_n^2}$$

- I_1 and K_1 are the modified Bessel functions of the first and second kind and order 1, L_1 is the modified Struve function of order 1
- The integration constants a_n^{II}, b_n^{II} have to be determined from interface conditions

In all Regions



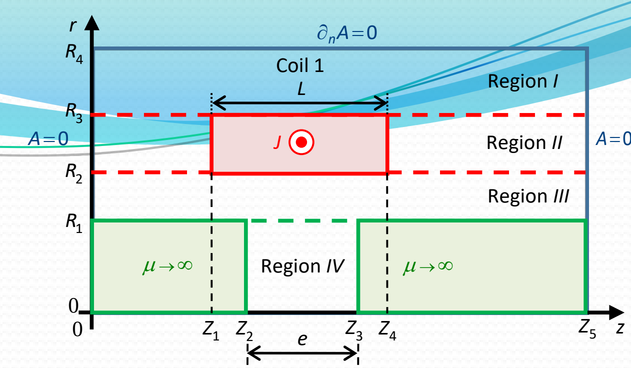
$$A_I(r,z)=\sum_{n=1}^{\infty} b_n^I\left(\frac{K_0(\alpha_n R_4)}{I_0(\alpha_n R_4)} I_1(\alpha_n r)+K_1(\alpha_n r)\right)\sin(\alpha_n z) \text{ with } \alpha_n = n\pi / Z_5$$

$$A_{II}(r,z)=\sum_{n=1}^{\infty}\left(a_n^{II} I_1(\alpha_n r)+b_n^{II} K_1(\alpha_n r)-C_n L_1(\alpha_n r)\right)\sin(\alpha_n z)$$

$$A_{III}(r,z)=\sum_{n=1}^{\infty}\left(a_n^{III} I_1(\alpha_n r)+b_n^{III} K_1(\alpha_n r)\right)\sin(\alpha_n z)$$

$$A_{IV}(r,z)=A_0 r+\sum_{k=1}^{\infty} a_k^{IV} I_1(\beta_k r)\cos\left(\beta_k (z-Z_2)\right) \text{ with } \beta_k = k\pi / (Z_3 - Z_2)$$

$$\text{with } A_0 = \left(R_1 (Z_3 - Z_2)\right)^{-1} \sum_{n=1}^{\infty} \alpha_n^{-1} \left(\begin{array}{l} \left(\cos(\alpha_n Z_2) - \cos(\alpha_n Z_3)\right) \times \\ \left(a_n^{III} I_1(\alpha_n R_1) + b_n^{III} K_1(\alpha_n R_1)\right) \end{array} \right)$$



Interface conditions

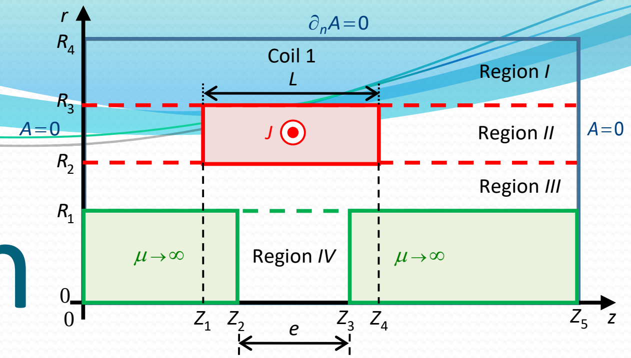
- At $r = R_3$

$$\begin{cases} A_{\mathrm{I}}(r=R_3,z)=A_{\mathrm{II}}(r=R_3,z) & \Leftarrow \text{ continuity of } A \\ \partial_r\left(r\,A_{\mathrm{I}}(r,z)\right)\Big|_{r=R_3}=\partial_r\left(r\,A_{\mathrm{II}}(r,z)\right)\Big|_{r=R_3} & \Leftarrow \text{ tangential continuity of } B \end{cases}$$

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & 0 & 0 & 0 \\ \eta_{21} & \eta_{22} & \eta_{23} & 0 & 0 & 0 \\ 0 & \eta_{32} & \eta_{33} & \eta_{34} & \eta_{35} & 0 \\ 0 & \eta_{42} & \eta_{43} & \eta_{44} & \eta_{45} & 0 \\ 0 & 0 & 0 & \eta_{54} & \eta_{55} & \eta_{56} \\ 0 & 0 & 0 & \eta_{64} & \eta_{65} & \eta_{66} \end{bmatrix} \begin{bmatrix} b_n^{I} \\ a_n^{II} \\ b_n^{II} \\ a_n^{III} \\ b_n^{III} \\ a_n^{IV} \end{bmatrix} = \begin{bmatrix} -C_n L_1\left(\alpha_n R_3\right) \\ -C_n L_0\left(\alpha_n R_3\right) \\ -C_n L_1\left(\alpha_n R_2\right) \\ -C_n L_0\left(\alpha_n R_2\right) \\ 0 \\ 0 \end{bmatrix}$$

$$\eta_{11}=\frac{K_0\left(\alpha_n R_4\right)}{I_0\left(\alpha_n R_4\right)}I_1\left(\alpha_n R_3\right)+K_1\left(\alpha_n R_3\right),\ \eta_{12}=-I_1\left(\alpha_n R_3\right),\eta_{13}=-K_1\left(\alpha_n R_3\right)\ldots$$

Inductance calculation



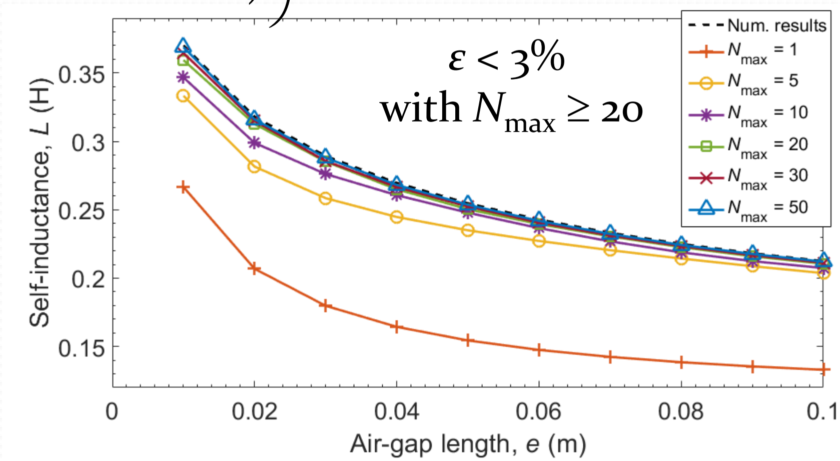
$$\varphi = \frac{2\pi}{S}\int_{Z_1}^{Z_4}\left(\int_{R_2}^{R_3} A_{II}\, r\, dr\right)dz,\ \Phi = N\varphi = L_{mc} I \text{ and } NI = JS$$

$$L_{mc} = 2\pi\frac{N^2}{JS^2}\sum_{n=1}^{N_{\max}}\left\{\frac{\pi}{2\alpha_n^{\,2}}\begin{pmatrix} a_n^{II}\left(R_3 U(R_3) - R_2 U(R_2)\right) \\ + b_n^{II}\left(R_3 V(R_3) - R_2 V(R_2)\right) \\ -\mu_0 J_n \dfrac{\alpha_n}{6\pi}\left(R_3^4 W(R_3) - R_2^4 W(R_2)\right)\end{pmatrix}\left(\cos(\alpha_n Z_1) - \cos(\alpha_n Z_4)\right)\right\}$$

$$U(r) = I_1(\alpha_n r)L_0(\alpha_n r) - I_0(\alpha_n r)L_1(\alpha_n r),$$

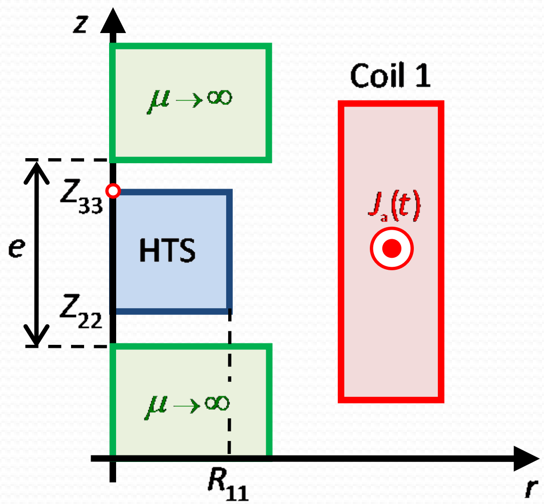
$$V(r) = K_1(\alpha_n r)L_0(\alpha_n r) + K_0(\alpha_n r)L_1(\alpha_n r),$$

$$W(r) = F\left(\{1,2\};\left\{\frac{3}{2},\frac{5}{2},3\right\};\frac{\alpha_n^2 r^2}{4}\right)$$



Coupling with Brandt's method

integral method



- The magnetic vector potential in the airgap is used as a source term
- The applied current density is now
  - pulsed $J_a(t) = J_{max}(t/\tau)\exp\left(1-(t/\tau)\right)$

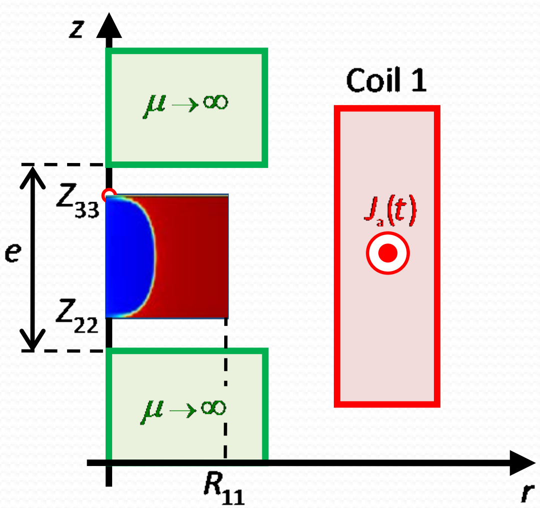
$$\frac{dJ(\mathrm{r}_2,t)}{dt} = -\mu_0^{-1}\int_0^{R_{11}}\int_{Z_{22}}^{Z_{33}} Q_{cyl}^{-1}(\mathrm{r}_2,\mathrm{r}_2')\left(E(J)+\frac{dA_{IV}(\mathrm{r}_2',t)}{dt}\right)dr'dz' \text{ with } \mathrm{r}_2=(r,z)$$

Coupling with Brandt's method

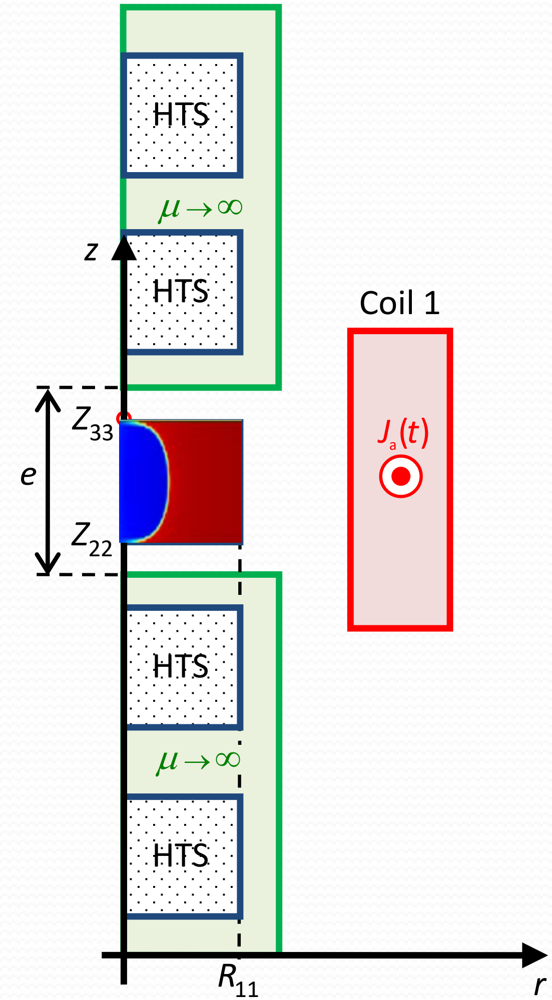
integral method

$$Q_{cyl}\left(r_2, r_2'\right) = \frac{1}{2\pi}\int_0^{\pi} \frac{r'\cos\varphi\, d\varphi}{\left((z-z')^2 + r^2 + r'^2 - 2rr'\cos\varphi\right)^{1/2}}$$

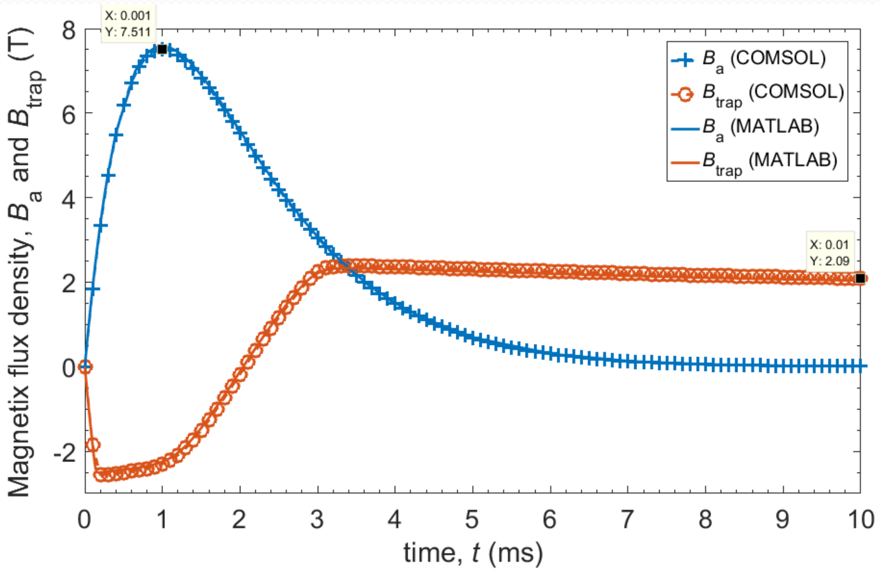
with $\varphi = \arctan(z/r)$



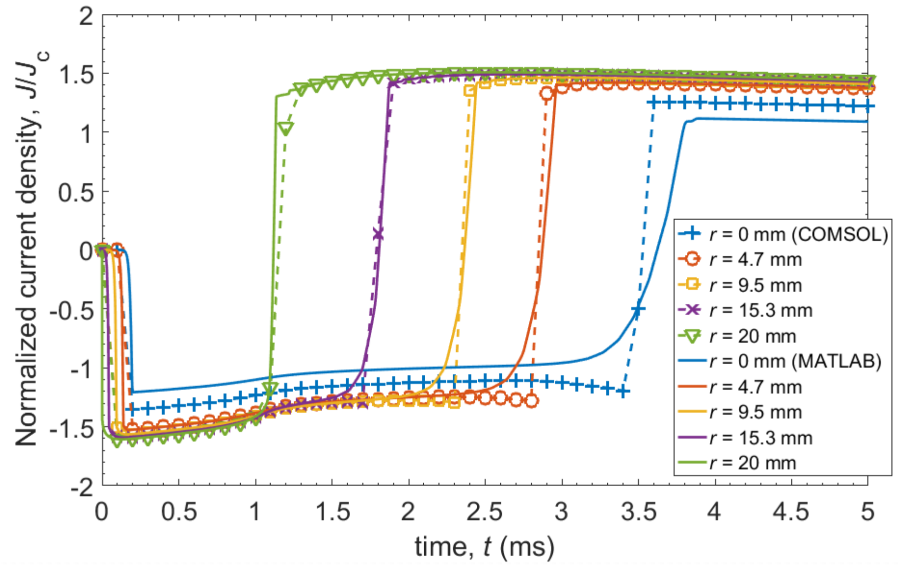
- Q_{cyl} is the kernel obtained by integrating the 3D Green function
- The ferromagnetic material influence is taken into account <u>by means of the method of images</u>



Results



Applied magnetic field B_a and trapped magnetic field B_{trap} on the top surface of the bulk HTS, at $r = 0$ and $z = Z_{33}$.

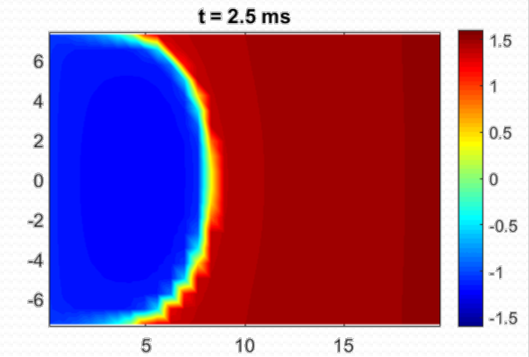


Normalized current density J / J_c in the middle plane of the bulk HTS for $r = \{0, 4.7, 9.5, 15.3, 20\}$ mm.

$$E(J) = (E_c / J_c)(|J| / J_c)^{n-1} J \text{ with } J_c = 100 \text{ A/mm}^2,\ n = 30,\ E_c = 1 \text{ µV/cm}$$

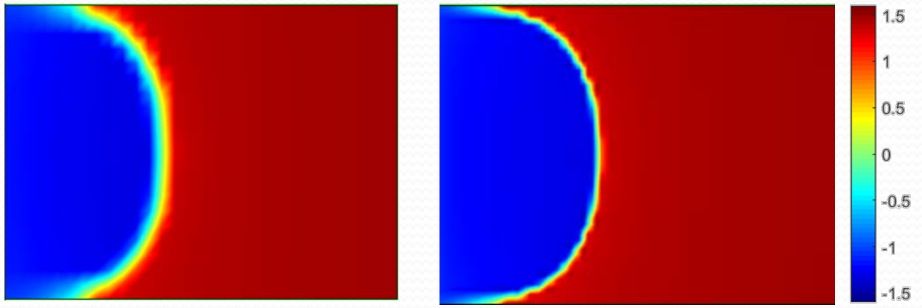
Computing times

Intel® Core™ i7-4600 CPU @ 2.10 GHz (Turbo Boost 2.70 GHz), 12 GB RAM (my laptop)



- Analytical calculation
  - < 1 s
- Construction of the topological matrices
  - 75 s
- Solving the system using ode23t (Matlab)
  - 642 s

- COMSOL 5.2a (using PDE)
  - 320 s with linear elements
  - 1896 s with quadratic elements



Future work

- Improve matrix conditioning
  - filtering, cut-off…
- Symbolic expression of the coefficients
  - a_n^I, b_n^I, a_n^{II}, b_n^{II}, a_n^{III}, b_n^{III} and a_n^{IV}
- Coupling with thermal equation
  - using finite difference method
- Adaptive mesh refinement
  - with the current density edge
- 3D problems (multiple bulks, magnetic lens…)
  - e.g. with "3D Simulation using the Fast Fourier Transform" from Leonid Prigozhin (https://arxiv.org/abs/1803.01346)

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Thank you for your attention!

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University of Lorraine - France

G.R.E.E.N.

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